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Correlation properties of fractal speckles in the Fresnel diffraction region

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This article is dedicated to Prof T Asakura

Three-dimensional correlation properties were studied theoretically and experimentally for fractal speckles produced in the Fresnel diffraction field. In the theoretical analysis, speckle patterns produced by a rough surface illuminated with coherent light with a power-law intensity distribution are assumed. It is shown that the intensity correlation in the longitudinal direction has different power law from the correlation across lateral planes. In the experiment, speckled-speckles were produced by a rough surface on which the speckle field due to a random fractal object is incident and their correlation properties were examined. The speckles observed in some lateral planes with different propagation distances did not exhibit a definite speckle size, having many intensity clusters with various sizes which tend to increase with an increase in the fractal dimension of the fractal object. The fractality across the lateral planes was confirmed by the existence of a power-law behavior in the intensity correlation, and was practically independent of the propagation distance. The longitudinal fractality was also revealed by a nearly power-law behavior in the longitudinal intensity correlation. It was shown that the longitudinal fractal dimension was larger than the lateral one for each dimension of the fractal object, indicating anisotropic fractality of the speckle field.

Keywords: Fractal speckle, Fresnel diffraction, Axial correlation, Anisotropic fractals, Correlation tail, Power law

1 Introduction

It is well known that various types of complex structures in nature are described by fractal geometry [1]. Typical examples are found in physical processes such as phase transitions, aggregations, and interface formations [1-3]. In the optical sensing of various quantities for such structures, we need to reveal how a light wave interacts with such structures having fractal property. For this reason, diffracted or scattered fields from objects with various fractal properties have been studied [4-11].

It would also be attractive to produce an optical field having a fractal property in a certain controllable manner in view of applications of the concept of fractals in the optical technology. It was shown theoretically that the generation of coherent optical field with a random fractal property is actually possible [12]. This prediction was subsequently verified by experiment [13], in which speckles produced in the Fourier transform plane of a ground glass plate were observed, where the glass plate was illuminated by speckles produced in the Fourier transform plane of a random fractal object. The observed speckles were shown to demonstrate a power-law behavior in their intensity correlation functions and, therefore, to have fractal property [13].

Another experiment of generating fractal speckles showed that a single scattering from a ground glass plate is enough to generate such speckles if the plate is illuminated by a power-law intensity distribution with a given exponent, and that such an illumination can be elaborated by means of a spatial light modulator [14,15]. Generation of much brighter fractal speckles was proposed on the basis of digital holographic approach [16,17]. Detailed theoretical analysis revealed that fractal speckles generated in the above configuration can

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be regarded to be multifractals [18]. Production of fractal speckles in an image plane was also discussed, and were actually performed experimentally by using an aperture having a power-law transmittance in the pupil plane of the imaging system [19]. In this report, an application of the image fractal speckles to the displacement measurements of diffusers was also demonstrated [19].

The present paper discusses the fractality of speckles produced in the Fresnel diffraction region as an extension of the previous theoretical analysis [12] and experiment [13] for fractal speckles produced in the Fraunhofer diffraction region. To this end, we study the lateral and longitudinal correlation functions of the speckle intensity distributions theoretically. Subsequently, speckles produced in some observation planes with different distances from ground glass plate in the Fresnel diffraction region are examined to confirm the theoretical predictions for the correlation properties expressing fractality of speckles.

Some results of the present work were presented in conferences and partly reported in their proceedings [20-23], and the present paper gives the full description and discussions of the subject.

2 Theoretical analysis

2.1 Fundamental properties of random mass fractals

Let us consider mass fractals for objects. Mass fractals are defined as the spatial mass distribution having the property that the total mass $M(u)$ included in a sphere of radius u obeys a power law

$$M(u) \propto u^{D}, \tag{1}$$

where D is the fractal dimension. As a property of mass fractals equivalent to the property of Eq (1), the correlation function $C(u)$ of the spatial mass distribution is also given by the power law

$$C(u) \propto u^{D-d}, \tag{2}$$

where d is the dimension of Euclidean space in which the fractal object is embedded.

A wave field scattered or diffracted by a fractal is called a diffractal. A typical example is the field observed in the Fraunhofer diffraction region of a random fractal aperture. It is known that such a field becomes speckles with an average intensity distribution following the power function

$$S(q) \propto q^{-D}, \tag{3}$$

where q is the radial coordinate of the observation plane.

Consider that an ordinary diffuser such as a ground glass plate is illuminated by light with an average intensity distribution following the power law of Eq (3). Then speckles of a specific kind are formed in the Fraunhofer diffraction region of the diffuser, and are expected to have a fractal property as shown theoretically by Uno *et al* [12], and experimentally by Uozumi *et al* [13]. They showed that the intensity correlation function of the speckle pattern exhibits an asymptotic behavior of the form expressed by

$$\mu(\Delta r) \propto \left(\frac{\Delta r}{z}\right)^{2(D-2)} \tag{4}$$

where Δr is the radial difference in the observation plane, and z is a propagation distance. This is due to a property of speckle phenomenon analogous to the van Cittert-Zernike theorem. That is, the intensity correlation function of the scattered field depends on the intensity distribution just behind the object.

2.2 Lateral correlation of fields diffracted by fractals in Fresnel region

Consider the optical geometry shown in Fig 1. Monochromatic light is incident on a rough surface, and the transmitted light is observed in the Fresnel diffraction region. If the surface is rough compared with the wavelength of the light, and if the illuminating spot is larger than the lateral correlation length of the irregularities of the surface, the speckle field $A(x,y,z)$ is a circular complex Gaussian random variable at each point (x,y,z). In this case, the normalized three-dimensional correlation function of intensity $I(x,y,z) = |A(x,y,z)|^2$ is given by

$$\mu(x_1,y_1,z_1;x_2,y_2,z_2) = \frac{\langle I(x_1,y_1,z_1)I(x_2,y_2,z_2)\rangle - \langle I(x_1,y_1,z_1)\rangle\langle I(x_2,y_2,z_2)\rangle}{\langle I(x_1,y_1,z_1)\rangle\langle I(x_2,y_2,z_2)\rangle}$$
$$= \frac{|J_A(x_1,y_1,z_1;x_2,y_2,z_2)|^2}{\langle I(x_1,y_1,z_1)\rangle\langle I(x_2,y_2,z_2)\rangle}, \quad (5)$$

where $J_A\ (x_1,y_1,z_1;x_2,y_2,z_2\ )$ is the mutual intensity of the field expressed by

$$J_A(x_1,y_1,z_1;x_2,y_2,z_2) = \langle A(x_1,y_1,z_1)\ A^*(x_2,y_2,z_2)\rangle. \quad (6)$$

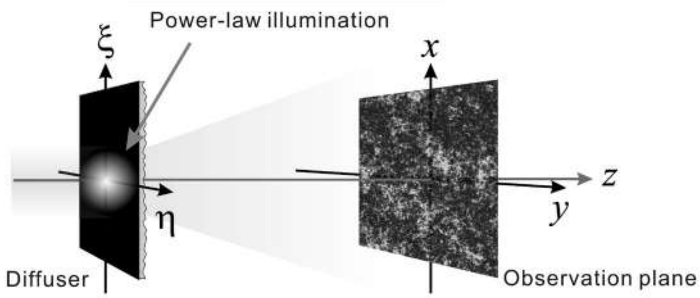


Fig 1. Optical system to generate fractal speckles in the Fresnel diffraction region.

The complex amplitude $A(x,y,z)$ of the speckle field at the observation point is given in terms of the amplitude transmittance $t(\xi,\eta)$ of the diffuser, pupil function $P(\xi,\eta)$ and proportional constant κ, being expressed under the Fresnel approximation by

$$A(x,y,z) = \frac{\kappa}{\lambda z}\iint_{-\infty}^{\infty} P(\xi,\eta)t(\xi,\eta)\exp\left\{i\frac{k}{\lambda z}[(\xi-x)^2+(\eta-y)^2]\right\}d\xi\, d\eta, \quad (7)$$

in which λ is the wavelength of the incident light and $k = 2\pi/\lambda$. Assuming that the illumination spot is much wider than the correlation length of the rough surface [24], we have

$$\langle t(\xi_1,\eta_1)\ t^*(\xi_2,\eta_2)\rangle = \delta(\xi_1-\xi_2,\ \eta_1-\eta_2). \quad (8)$$

To consider the lateral correlation property of the scattered field, let us limit the coordinate system of the correlation of Eq (5) to xy plane and assume the stationarity of the field and the relation of Eq (8). Then, the lateral two-dimensional intensity correlation function can be expressed by

$$\mu(\Delta x,\Delta y) = \frac{\langle I(x_1,y_1)\ I(x_2,y_2)\rangle - \langle I(x_1,y_1)\rangle\langle I(x_2,y_2)\rangle}{\langle I(x_1,y_1)\rangle\langle I(x_2,y_2)\rangle}$$
$$= \frac{1}{S^2}\left|\iint |\mathrm{P}(\xi,\eta)|^2\exp\left[-i\frac{2\pi}{\lambda z}(\xi\Delta x+\eta\Delta y)\right]d\xi d\eta\right|^2, \quad (9)$$

where $\Delta x = x_1 - x_2$, $\Delta y = y_1 - y_2$ and

$$S = \iint |P(\xi,\eta)|^2\ d\xi d\eta. \quad (10)$$

Then we consider that incident light obeys the power-law as is the case that illuminating field is a diffractal having the property of Eq (3). However, power function of Eq (3) diverges as q approaches the origin. To avoid this problem, the pupil function is approximated by the Fisher-Burford formula [25]

$$|P(\xi,\eta)|^2 = (1+\alpha^2\rho^2)^{-D/2}, \quad (11)$$

where α is constant to control the behavior the above function around the origin and $\rho = \sqrt{\xi^2+\eta^2}$. Substituting Eq (11) into Eq (9), we obtain

$$\mu(\Delta r) = \frac{4\pi^2}{\alpha^4 S^2} \frac{\left(\frac{k\Delta r}{\alpha z}\right)^{D-2} K^2_{D/2-1}\left(\frac{k\Delta r}{\alpha z}\right)}{2^{D-2}\,\Gamma^2(D/2)} \quad \text{for } 1 < D < 3, \tag{12}$$

where $\Delta r = \sqrt{\Delta x^2 + \Delta y^2}$, while $K_v(x)$ is the modified Bessel function of the second kind of order v and $\Gamma(\cdot)$ is gamma function. S is evaluated from Eqs (10) and (11), with the result

$$S = \begin{cases} \infty & \text{for } 0 < D < 2, \\ \dfrac{2\pi}{\alpha^2(D-2)} & \text{for } 2 < D < 3. \end{cases} \tag{13}$$

Since S diverges for $0 < D < 2$ and $(\Delta r/z)^{D-2}$ also diverges for $0 < D < 2$ as $(\Delta r/z) \to 0$, we treat the function $\mu(\Delta r)$ in its proportional behavior rather than in its absolute value. The asymptotic behavior of Eq(13) in the range of $\Delta r/z \ll 1$ is therefore given by

$$\mu(\Delta r) \propto \frac{4\pi^2}{\alpha^2 S^2} \begin{cases} \dfrac{\left(\frac{k\Delta r}{\alpha z}\right)^{2(D-2)}}{2^{D-2}\Gamma^2(D/2)} & \text{for } 1 < D < 2, \\ \dfrac{1}{2^{D-2}\Gamma^2(D/2)} & \text{for } 2 < D < 3, \end{cases} \tag{14}$$

because the function $K_{D/2-1}(\cdot)$ asymptotically behaves as

$$K_{D/2-1}(v) \propto \begin{cases} v^{D/2-1} & \text{for } 1 < D < 2, \\ v^{1-D/2} & \text{for } 2 < D < 3, \end{cases} \tag{15}$$

as v approaches zero. Thus, for $1 < D < 2$, lateral two-dimensional intensity correlation function in Fresnel diffraction region is given by

$$\mu(\Delta r) \propto \left(\frac{k\Delta r}{\alpha z}\right)^{2(D-2)}. \tag{16}$$

It is noted that the above results of Eqs (12) – (16) are practically analogous to the results derived for fractal speckles generated in the Fraunhofer diffraction region of a diffuser [12]. Therefore, it is concluded that the type of the lateral correlation function in the Fresnel diffraction region is the same as that in Fraunhofer diffraction region, which implies that lateral two-dimensional intensity distribution of speckles in the Fresnel region has the same fractal property as that in the Fraunhofer region.

2.3 Longitudinal correlation property in Fresnel diffraction region

With reference to Fig 1 for optical configuration again, we discuss the longitudinal correlation on the field scattered by a fractal object in the Fresnel diffraction region. Starting with Eqs (5) – (8), the mutual intensity $J_A(x_1,y_1,z_1;x_2,y_2,z_2)$ can be expressed by

$$J_A(\Delta x',\Delta y',\Delta z) = \frac{\kappa^2}{\lambda^2 z(z+\Delta z)} \times \iint_{-\infty}^{\infty} |P(\xi,\eta)|^2 \exp\left[-i\frac{k}{2}\frac{\Delta z}{z(z+\Delta z)}(\xi^2+\eta^2)\right] \exp[-ik(\Delta x'\xi + \Delta y'\eta)]d\xi d\eta, \tag{17}$$

where

$$\Delta x' = \frac{x_1}{z_1} - \frac{x_2}{z_2}, \qquad \Delta y' = \frac{y_1}{z_1} - \frac{y_2}{z_2}, \qquad \Delta z = z_1 - z_2. \tag{18}$$

Note that the normalization of x and y by in Eq (18) takes into account the lateral expansion of the diffraction pattern with the propagation distance z.

To progress further, we consider the following two conditions: (i) no lateral shifts, i.e., $\Delta x' = \Delta y' = 0$, (ii) illumination light with a power-law intensity distribution given by Eq (11). Under these conditions, we obtain;

$$\mu(\Delta z) = \frac{\pi^2}{\alpha^4 S^2}\Bigg| -i\Gamma(1-D/2)\, e^{iD\pi/4}\, s^{D/2-1} - \frac{1}{2-D}\left[{}_1F_1\left(1-\frac{D}{2};\, 2-\frac{D}{2};\, is\right) + {}_1F_1\left(1-\frac{D}{2};\, 2-\frac{D}{2};\, -is\right)\right] - \frac{i}{2-D}\left[{}_1F_1\left(1-\frac{D}{2};\, 2-\frac{D}{2};\, is\right) - {}_1F_1\left(1-\frac{D}{2};\, 2-\frac{D}{2};\, -is\right)\right]\Bigg|^2 \text{ for } 0 < D < 2, \tag{19}$$

where

$$s = \frac{k}{2\alpha^2}\,\frac{\Delta z}{z(z+\Delta z)}, \tag{20}$$

and ${}_1F_1\,(\cdot;\,\cdot;\,\cdot)$ is the confluent hyper geometric function. Since $s^{D/2-1}$ diverges for $0 < D < 2$ as $s \to 0$, we assume again that the function $\mu(\Delta z)$ makes sense only in its proportional behavior. Then the asymptotic behavior of Eq(19) for $s \ll 1$ is given by

$$\mu(\Delta z) \propto \frac{\pi^2}{\alpha^4 S^2}\left| -is^{D/2-1}\,\Gamma\left(1-\frac{D}{2}\right)\exp\left(i\frac{D\pi}{4}\right) - \frac{2}{2-D} - \frac{2s}{4-D}\right|^2. \tag{21}$$

Equation (21) is still a little complicated. For $D < 2$ and sufficiently small S, however, the first term in the absolute symbol will be dominant over the others, which would give rise to the relation of

$$\mu(\Delta z) \propto \Delta z^{D-2}. \tag{22}$$

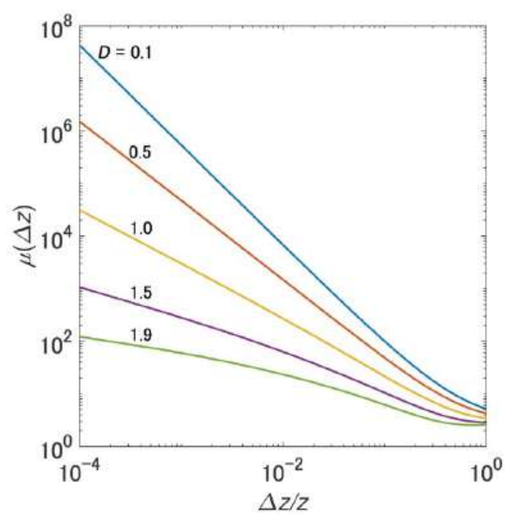


Fig 2. Theoretical curves of the intensity correlation function $\mu(\Delta z)$ of Eq (21) for fractal dimensions of D = 0.1, 0.5, 1.0, 1.5 and 1.9.

The functional behavior of the intensity correlation $\mu(\Delta z)$ of Eq(21) is numerically evaluated and shown in Fig 2 for several values of $0.1 \le D \le 1.9$. The abscissa of the figure is normalized by z. For small $\Delta z/z$, the function $\mu(\Delta z)$ seems to be linear in the logarithmical representation, implying the fractal behavior of speckles in this small range. To confirm further this linearity in the logarithmic representation, the derivative $d\log\mu(\Delta z)/d\log(\Delta z)$ is calculated and the result is shown in Fig 3. In this figure, as $\log\Delta z$ decreases, the derivative appears to asymptotically approach constant values b = –1.9, –1.5, –1.0, –0.5 and –0.1 shown by dashed lines for the dimension D = 0.1, 0.5, 1.0, 1.5 and 1.9, respectively, indicating the relation of $b = D - 2$. These results confirm that, as Δz decreases, the intensity correlation function asymptotically obeys the power-law of Eq (22). Thus the speckle has a fractal behavior in its longitudinal direction with its exponent dependent on the fractal dimension D of the object. It is also noted that,with reference to Eq (16), the lateral and longitudinal correlations obey different power-laws and, hence, the fractality of speckle intensities in the Fresnel diffraction region is anisotropic.

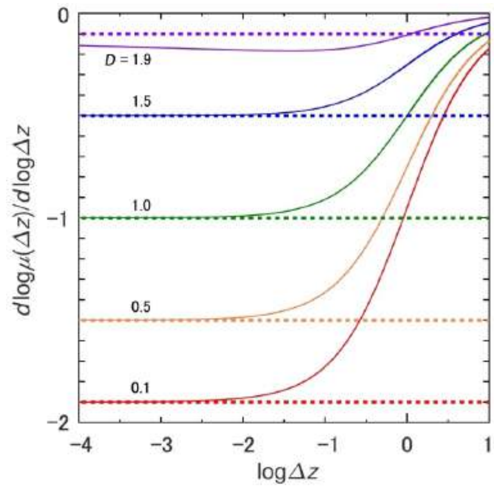


Fig 3. Derivative $d\log\mu(\Delta z)/d\log\Delta z$ of the intensity correlation function $\mu(\Delta z)$ of Eq (21) for fractal dimensions of $D = 0.1$, 0.5, 1.0, 1.5 and 1.9.

3 Experiment and discussions

3.1 Lateral correlation

Experimental set-up is shown in Fig 4. A Gaussian beam emerging from a He-Ne laser was suitably expanded and collimated by lenses L_1 and L_2, and brought to illuminate a random fractal object placed at the front focal plane P_1 of lens L_3. The field scattered from the aperture with fractal property has the average intensity distribution following the power function, as given by Eq (3), in the Fraunhofer diffraction plane P_2,which is the back focal plane of L_3.This field was incident on a ground glass plate placed in plane P_2. Then, the speckle pattern formed in the observation plane P_3 at z = 7.5, 15, 30 cm in the Fresnel diffraction region of the ground glass plate was recorded by a CCD camera. Speckles were also recorded at the back focal plane of a lens L_4, which is not shown in Fig 4, with focal length f = 20 cm placed behind the diffuser to obtain speckles in the Fraunhofer diffraction region for comparison. Recorded speckle intensities were stored in a computer as discrete data with 256 levels and served for the statistical analysis. Fractal objects used in the experiment were generated by Weierstrass function as shown in Fig 5 [26]. Three random fractal apertures with different fractal dimensions were printed out by a laser printer, and subsequently photographed on film, which were used as the fractal apertures.

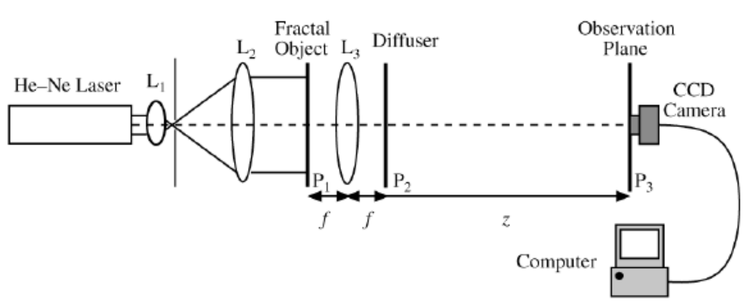


Fig 4. Experimental set-up for producing fractal speckles in Fresnel diffraction region of a diffuser.

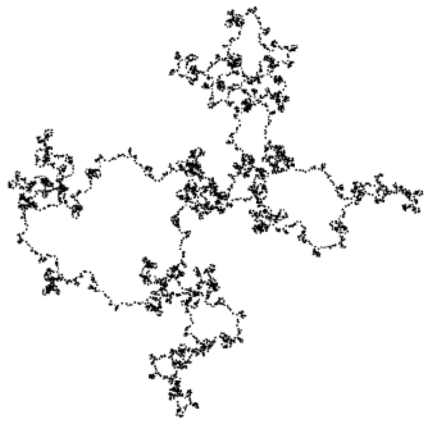


Fig 5. Fractal object of $D = 1.49$.

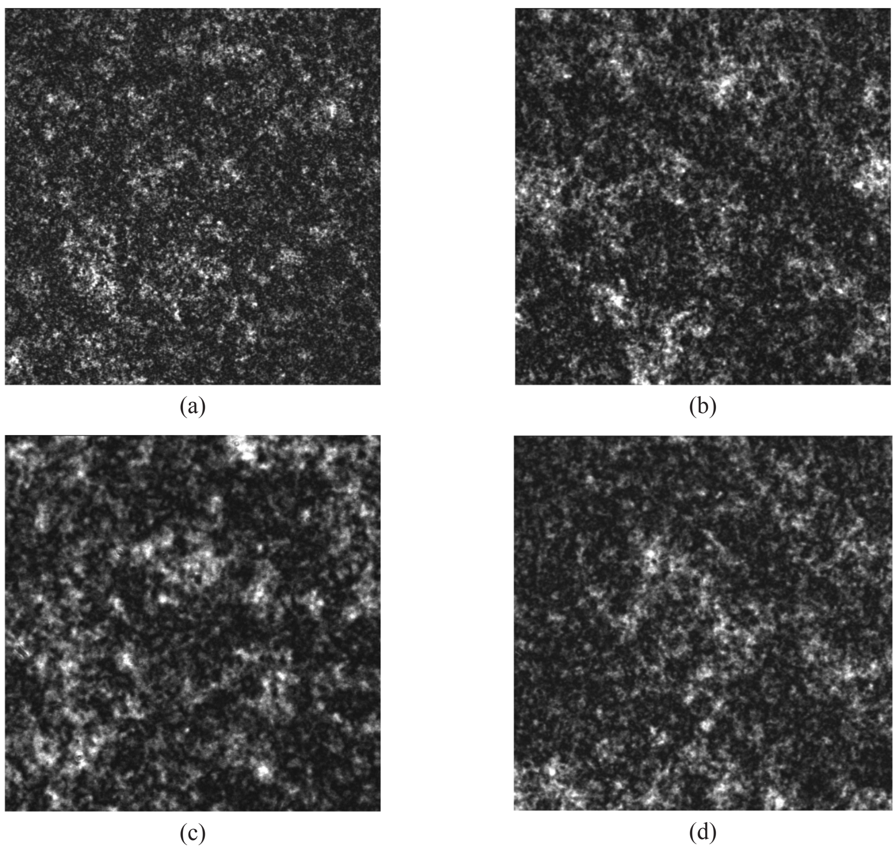


Fig 6. Speckle patterns produced in the observation plane in the Fresnel diffraction region at (a) $z = 7.5$, (b) 15, (c) 30 cm, and (d) in the Fraunhofer diffraction region of the diffuser by using the fractal object of $D = 1.49$ to generate a power-law intensity distribution for illumination of the diffuser.

Figures 6(a)–(d) show the speckle patterns observed at z = (a) 7.5, (b) 15,(c) 30 cm, and (d) in the Fraunhofer diffraction region, generated with the fractal aperture with $D = 1.49$. As seen in this figure, observed speckle patterns have lumps or clusters of intensity with different scales with a remarkable feature that they look similar in a statistical sense, though the pattern looks expanded as z increases in the Fresnel diffraction region.

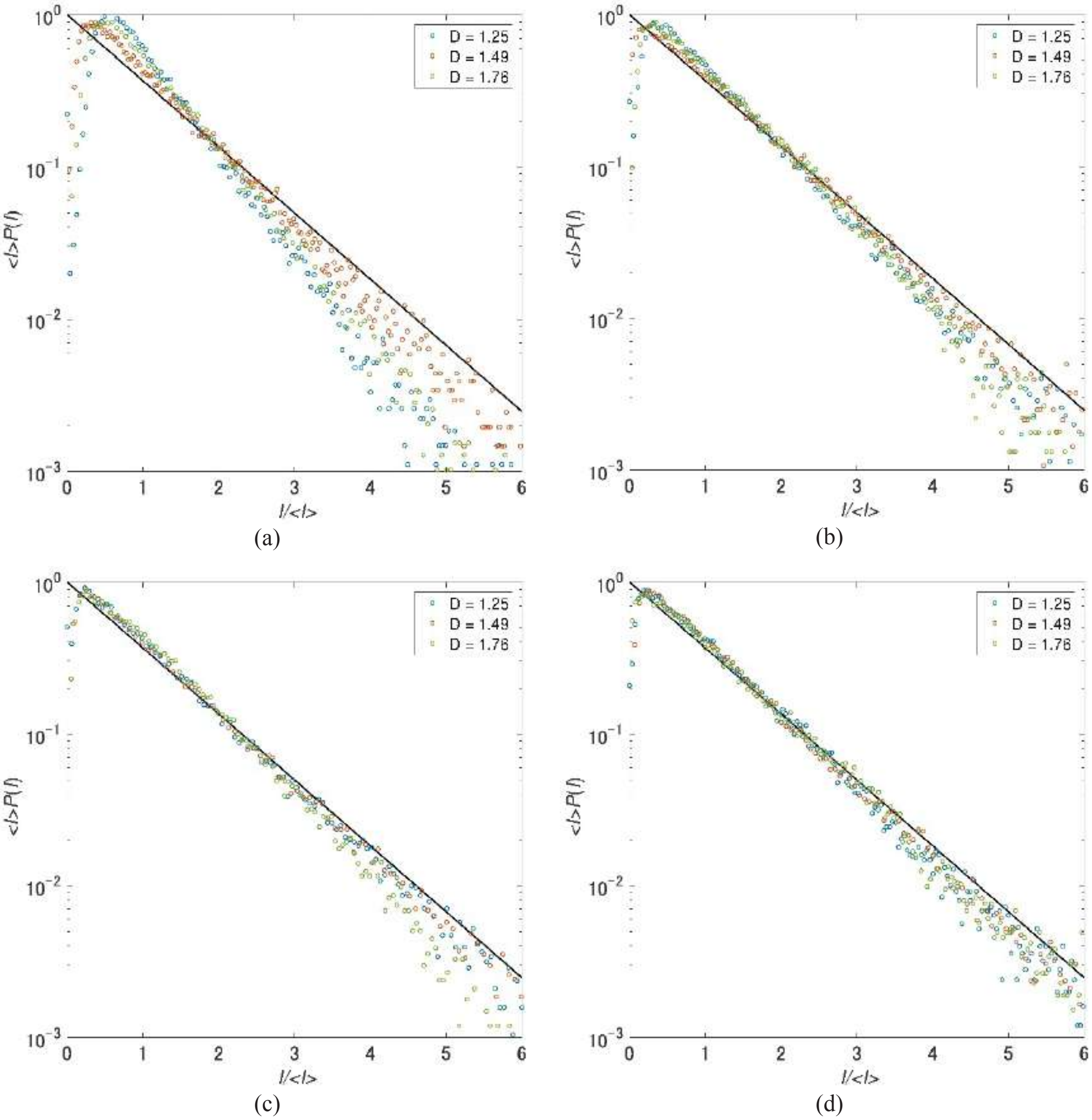
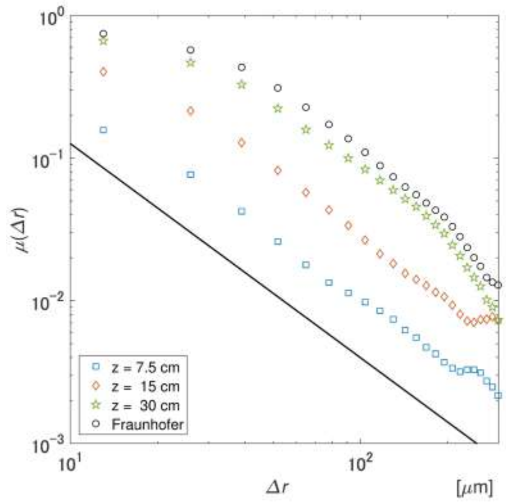
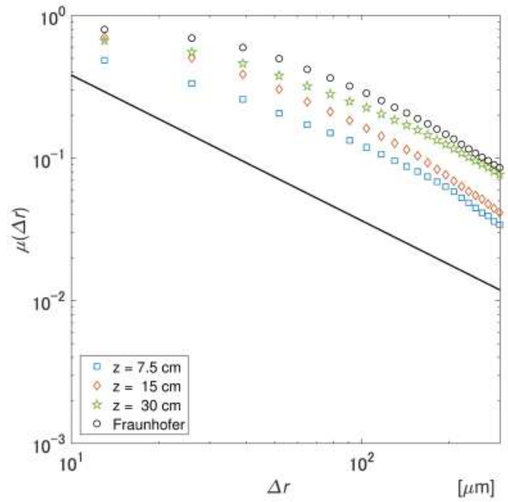


Fig 7. Experimental probability density functions calculated from speckle intensities observed at z = (a)7.5, (b)15 and (c) 30 cm, and (d) in the Fraunhofer diffraction region. Fractal dimension of the employed objects are D = 1.25, 1.49 and 1.76.

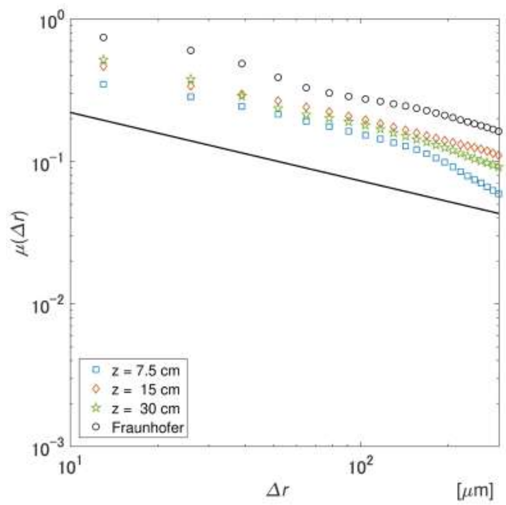
In the theoretical analysis in Sec 2, we assumed that the complex amplitude $A(x,y)$ is a circular complex Gaussian random variable. It is known that probability density of speckle intensity of this type obeys the negative exponential distribution [24]. Therefore, the probability densities of the observed speckles were calculated and are shown in Figs 7(a)–(d) for z = (a)7.5, (b)15, (c)30 cm and (d) the Fraunhofer diffraction region. Figure 7 shows that, as the propagation distance decreases, the probability density tends to deviate from the theoretical negative exponential function, which is represented by solid lines. As is seen in Fig 6 and also indicated by Eq (18), the diffraction pattern decreases in its size as z decreases, and it follows that the minimum size of speckles falls below the pixel size of the image sensor. This causes integration of speckle intensity over each pixel, giving rise to the deviation of probability density from the negative exponential as seen in Fig 7.



(a)



(b)



(c)

Fig 8. Intensity correlation functions $\mu(\Delta r)$ of observation speckles produced by the random fractal objects of D = (a)1.25, (b)1.49 and (c) 1.76 and observation plane z = (□) 7.5, (◇)15, (☆) 30 cm, and (○) in the Fraunhofer diffraction region.

To examine, if these in-plane speckles have fractal property, two-dimensional autocorrelation functions of the speckle intensities are evaluated. The autocorrelation functions were calculated by combining the following two averaging operations; (i) angular average of a two-dimensional correlation of a speckle pattern, (ii) average over ten correlation functions of speckles obtained for different illumination spots on the diffuser. These two-fold averaging process ensures efficient suppression of statistical fluctuations. The results are shown logarithmically in Fig 8. In Figs 8(a)–(c), fractal dimension is D = (a)1.25, (b)1.49 and (c)1.76 for three objects used for generating fractal speckle, and the observation planes are at the distance of $z = 7.5$, 15, 30 cm and in the Fraunhofer diffraction region. The solid lines represent power function with the exponent given by Eq (16). It is seen in this figure that the intensity correlation functions are approximately linear for $\Delta r < 200$ μm in a logarithmical plot and have slopes close to the theoretical one. This confirms that the lateral two-dimensional correlation function is nearly a power function represented by Eq (16) in this range of Δr, indicating that the lateral speckles can be regarded as fractals at least for the first approximation.

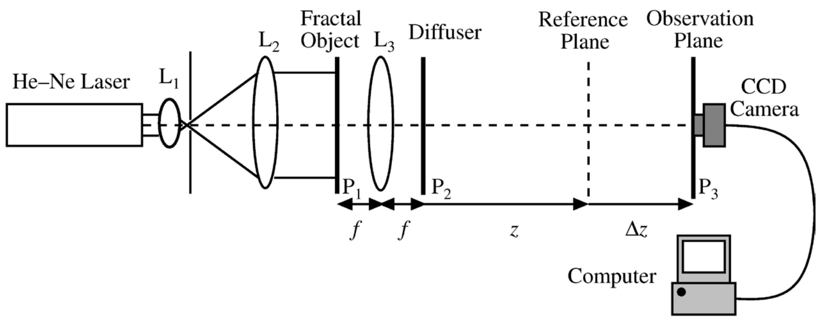
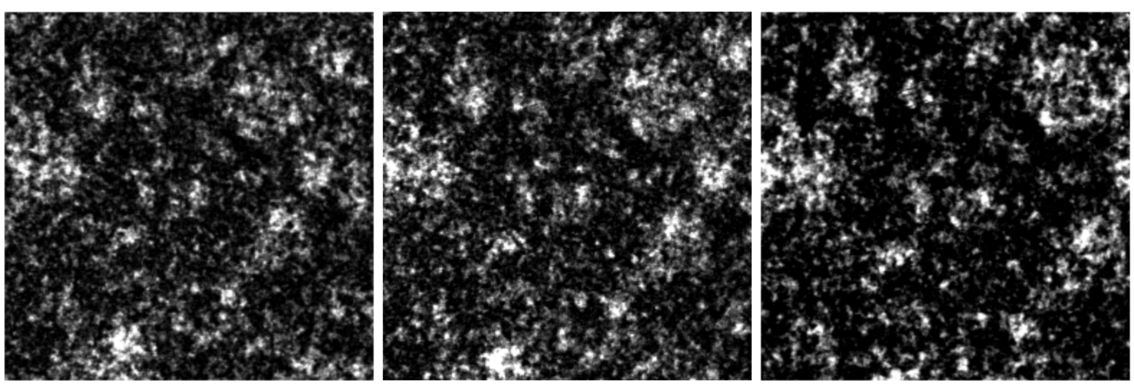


Fig 9. Experimental set-up for observing fractal speckles produced at different distances Δz from a reference plane at $z = 10$ cm.

It is to be noted that D is the fractal dimension of the object placed in P_1 in Fig 4, and is not a fractal dimension of speckles observed. Since the intensity is a non-negative quantity, it would be possible to consider the spatial intensity distribution as a spatial mass distribution. Then the intensity of fractal speckles is regarded as a mass fractal. Indeed, we discussed the fractality of speckles from this viewpoint on the basis of Eq (2). Therefore, if we let D_{sl} denote the fractal dimension of the lateral speckle intensity distribution, the exponent of the intensity correlation function should be $D_{sl} - d$, with d being the Euclidean dimension. By combining this relation with Eq (16) and noting $d = 2$ for the two-dimensional lateral correlation, we obtain a relation of $D_{sl} = 2D - 2$. Hence, the fractal dimension of the lateral two-dimensional intensity distribution is $D_{sl} = 0.5$, 0.98 and 1.52 for $D = 1.25$, 1.5 and 1.76, respectively.



(a) (b) (c)

Fig 10. Speckle patterns observed at three different distances Δz = (a) 7.5, (b) 15, and (c) 30 mm from the reference plane at $z = 10$ cm from the diffuser.

3.2 Longitudinal correlation

The experimental set-up for investigating the longitudinal correlation of speckles is shown in Fig 9. The fractal object, diffuser, and illumination optics are the same with Fig 4, while speckles were recorded at different longitudinal distances $z + \Delta z$, where z = 10 cm stands for a reference plane and Δz is an additional distance from the reference plane. Therefore, the longitudinal correlation property can be obtained by the central peak values of the two-dimensional cross-correlation of speckles in the reference plane and each of observation planes at various Δz from the reference plane.

Three speckle patterns observed at successive distances of Δz = (a) 7.5, (b) 15, and (c) 30 mm are shown in Figs 10(a)–(c), respectively. This sequence of patterns does not seem to change largely with Δz, though the pattern looks expanded gradually. This implies that the axial correlation has a long correlation tail.

A sequence of two-dimensional cross-correlation functions were calculated from speckle patterns observed at different distances of $1 < \Delta z < 25$ cm. In this calculation, the expansion of the diffraction patterns in the Fresnel diffraction region was compensated using the relations of Eq(18). A logarithmical plot of the peak correlation values of the obtained cross-correlations given by each speckle patterns are shown in Fig 11. It is seen in this figure that, for $1 < \Delta z \lesssim 20$ mm, the correlation peak value decreases almost linearly with the slope of $2 - D$, which is represented by solid lines. Therefore, it is confirmed experimentally that the longitudinal correlation obeys the power function predicted by Eq (22).

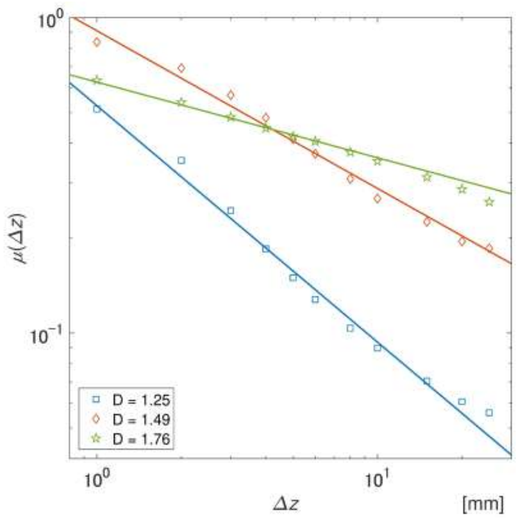


Fig 11. Longitudinal intensity correlation functions obtained from the central peak of the cross-correlation functions of several pairs of speckle patterns at the distances z and $z + \Delta z$.

It is noted that Euclidean dimension d is unity for the present discussion of the longitudinal correlation. Therefore, combining the exponent $D - 2$ of Eq (22) and the exponent $D_{sa} - d$ of Eq (16), with D_{sa} denoting longitudinal fractal dimension of speckles, we obtain a relation of $D_{sa} = D - 1$. Hence, we have D_{sa} = 0.25, 0.49, and 0.76 for D = 1.25, 1.49, and 1.76, respectively. It follows from the lateral fractal dimension $D_{sl} = 2D - 2$ that we have $D_{sl} = 2D_{sa}$ for the speckles in the Fresnel diffraction region. It is noted, however, that this factor of two is not attributed to the difference in the Euclidean dimension d. If we consider a one-dimensional cross section of the lateral two-dimensional speckle intensity distribution, its fractal dimension does not reduce to D_{sa}, but is given by for $D_{sl} - 1 = 2D - 3$ for $D_{sl} > 1$ and 0 for $D_{sl} < 1$ [3]. Therefore, the fractality of speckles produced in the Fresnel diffraction region is considered to be anisotropic.

4 Conclusion

Correlation properties of speckles produced in the Fresnel diffraction region of a diffuser illuminated by diffractals of random fractal objects was investigated theoretically and experimentally. The correlation function of speckles in the Fresnel diffraction region was derived theoretically for lateral two-dimensional and longitudinal one-dimensional directions and the results were numerically evaluated. It was revealed from the theoretical analysis that the lateral intensity correlation function exhibits a power-law behavior having an exponent of $2D - 2$, with D being a fractal dimension of the object generating the diffractal to illuminate the diffuser. This exponent is the same with that for fractal speckles in the Fraunhofer diffraction region and, therefore, the lateral intensity distribution of speckles in the Fresnel region possesses the same fractality with that in the Fraunhofer region, having the fractal dimension $D_{sl} = 2D - 2$.

Analysis with respect to longitudinal intensity correlation function of the speckles in the Fresnel diffraction region also yielded a power function as an asymptotic behavior for small distance in the axial direction, implying an longitudinal fractality of the speckles. However, the exponent of the power function in the longitudinal direction is $D - 2$, being different from that for the lateral two-dimensional one. This gives the axial fractal dimension $D_{sa} = D - 1$, which is different from the lateral dimension, meaning anisotropic fractality of the speckle intensity distribution in the Fresnel diffraction region.

Fractal speckles discussed in the theoretical analysis were generated experimentally using a random fractal object as a scatterer for generating a diffractal field to illuminate the diffuser. From the speckle intensity detected in the Fresnel diffraction region, different power-law behaviors were confirmed between the lateral two-dimensional and longitudinal one-dimensional directions predicted by the theory.

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Appendix

Derivation of Eqs (17), (19) and (21)

By substituting Eq (7) into Eq (6) and using the assumption of Eq (8), the mutual intensity J_A is given by

$$J_A(\Delta x',\Delta y',\Delta z) = \frac{\kappa^2}{\lambda^2 z_1 z_2} \iint_{-\infty}^{\infty} |P(\xi,\eta)|^2 \times \exp\left\{-i\frac{k}{2z_1}\left[(x_1-\xi)^2+(y_1-\eta)^2\right]\right\} \exp\left\{-i\frac{k}{2z_2}\left[(x_2-\xi)^2+(y_2-\eta)^2\right]\right\} d\xi d\eta. \quad \text{(A1)}$$

After some manipulations, with the replacement of and $\Delta x' = x_1/z_1 - x_2/z_2$, $\Delta y' = y_1/z_1 - y_2/z_2$, and $\Delta z = z_1 - z_2$, Eq (A1) becomes

$$J_A(\Delta x',\Delta y',\Delta z) = \frac{\kappa^2}{\lambda^2 z(z+\Delta z)} \iint_{-\infty}^{\infty} |P(\xi,\eta)|^2 \times \exp\left[-i\frac{k}{2}\frac{\Delta z}{z(z+\Delta z)}(\xi^2+\eta^2)\right] \exp\left[-ik(\Delta x'\xi+\Delta y'\eta)\right] d\xi d\eta, \quad \text{(A2)}$$

which is Eq (17).

If we consider the case of $\Delta x' = \Delta y' = 0$ and assume Eq (11) for the illumination, Eq (A2) is simplified to, with the constant in front of the integral replaced by c,

$$J_A(\Delta z) = c\int_0^{2\pi}\int_0^{\infty}(1+\alpha^2\rho^2)^{-D/2}\exp\left[-i\frac{k}{2}\frac{\Delta z}{z(z+\Delta z)}(\xi^2+\eta^2)\right]\rho d\rho d\theta, \quad \text{(A3)}$$

which reduces, after some manipulations, to

$$J_A(\Delta z) = c\frac{c'\pi}{\alpha^2}\int_1^{\infty} u^{-D/2}\exp(-isu)du, \quad \text{(A4)}$$

where s is given by Eq (20) and c' is another constant. Using the following formulae,

$$\int_0^{\infty} x^{-v}\cos(xy)dx = y^{v-1}\,\Gamma(1-v)\sin\frac{v\pi}{2},\; 0<\mathrm{Re}v<1,$$

$$\int_0^{\infty} x^{-v}\sin(xy)dx = y^{v-1}\,\Gamma(1-v)\cos\frac{v\pi}{2},\; 0<\mathrm{Re}v<2,$$

$$\int_0^{1} x^{v-1}\cos(xy)dx = \frac{1}{2v}\left[{}_1F_1(v;v+1;iy) + {}_1F_1(v;v+1;-iy)\right],\; 0<\mathrm{Re}v,$$

$$\int_0^{1} x^{v-1}\sin(xy)dx = \frac{1}{2v}\left[{}_1F_1(v;v+1;iy) - {}_1F_1(v;v+1;-iy)\right],\; -1<\mathrm{Re}v,$$

Eq (A4) is evaluated to be

$$J_A(\Delta z) = \frac{c'\pi}{\alpha^2}\left\{-is^{D/2-1}\,\Gamma\left(1-\frac{D}{2}\right)\exp\left(i\frac{D\pi}{4}\right) - \frac{1}{2-D}\left[{}_1F_1\left(1-\frac{D}{2};2-\frac{D}{2};is\right) + i\,{}_1F_1\left(1-\frac{D}{2};2-\frac{D}{2};-is\right)\right] + \frac{i}{2-D}\left[{}_1F_1\left(1-\frac{D}{2};2-\frac{D}{2};is\right) - i\,{}_1F_1\left(1-\frac{D}{2};2-\frac{D}{2};-is\right)\right]\right\} \text{ for } 0<D<2, \quad \text{(A5)}$$

which gives Eq (19) after substituted into

$$\mu(\Delta z) = \left| \frac{J_A(0,0,\Delta z)}{J_A(0,0,0)} \right|^2. \tag{A6}$$

When $s \ll 1$, terms of higher order in the series expansion of the function ${}_1F_1\,(\cdot\,;\cdot\,;\cdot)$

$$ {}_1F_1\,(\alpha;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{\Gamma(\gamma+n)} \frac{z^n}{n!} \tag{A7}$$

can be ignored and Eq (A5) finally reduces to

$$J_A\,(\Delta z) = \frac{c'\pi}{\alpha^2} \left[-is^{D/2-1}\,\Gamma\!\left(1-\frac{D}{2}\right) \exp\left(i\,\frac{D\pi}{4}\right) - \frac{2}{2-D} - \frac{2s}{4-D} \right] \quad \text{for } 0 < D < 2, \tag{A8}$$

which gives Eq (21) after substituted into Eq (A6).

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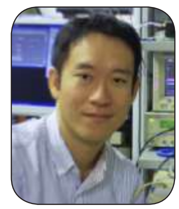

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