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Partial coherence in transport of intensity phase imaging

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Transport of intensity phase imaging is a method of propagation–based quantitative phase retrieval that utilizes short propagation distances to reduce the phase retrieval problem to a deterministic, linear equation. As usually formulated, the phase of a coherent field can be recovered by taking three or more intensity measurements through focus. However, for partially coherent light the field has no analogue to phase. Instead, one can retrieve the thickness of samples that are illuminated by partially coherent light provided the coherence effects are taken into account. Here, we review two methods to do so for general states of coherence where additional measurements are required to characterize both the illumination and the sample and for Kohler illumination where partial coherence manifests as a blurring of the recovered phase.

Keywords: Partial coherence, Phase retrieval, Transport of intensity

1 Introduction

Visible light is an electromagnetic field oscillating in the range of 430–770 THz. A measurement of the electric field can be used to produce images of a sample. Imaging detectors operate many orders of magnitude slower than the oscillations of the field which results in the recording only of time–averaged power or intensity of the electric field at the detector pixels, such that the relative phase delay of the field at different points on the detector is not measured. The intensity contains direct information about the absorption or scattering of light by the sample. However, phase differences in the electric field can encode information about the thickness or surface profile of a sample. Retrieving thickness/depth information can be useful in a variety of cases ranging from quantitative thickness estimation of biological samples in microscopy (where absorption differences may be quite small), to surface metrology. The problem of phase retrieval can be broadly thought of as a class of techniques to recover phase differences in the field from intensity measurements.

Interferometry is a widely–used technique to recover phase that typically relies on the precise alignment of two interfering beams of light. By measuring the interference pattern produced at the detector, and from knowledge of the illumination, the phase introduced by the sample can be reconstructed. Interferometry can produce precise phase results but does present some challenges. The interfering beams must be maintained in alignment to within a wavelength (several hundred of nanometers for visible light) during the measurement process to avoid shifts in the interference pattern. Moreover, phase unwrapping is required due to the fact that any integral number of 2π radians may be added to the sample's phase at any point without changing the interference pattern measured. Interferometry typically requires a high degree of spatial and temporal coherence of light, which may not always be desirable since it can lead to diffraction artifacts in the acquired images.

Propagation–based phase imaging relies on the fact that the propagation involves the self–interference of a beam of light with itself. The intensity of the beam measured after propagation relies not only on the

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initial intensity of the beam, but also on the initial phase of the beam. Unlike interferometry, this requires only the free–space propagation of light, alleviating the requirement of precise alignment and stability. This can be extremely useful for regimes beyond visible light, such as x–ray [1, 2], neutron beam [3] and electron beam [4, 5] imaging where high quality optical components may not be easy to produce and stable alignment to within a wavelength is extremely challenging. Additionally, phase unwrapping is typically not a concern due to the propagation models employed. Additionally, depending on the system employed, a high degree of coherence (spatial or temporal) may not be necessary. As a general class, propagation–based phase imaging techniques rely on a detailed model of propagation that mathematically describes the measured intensity after propagation as a function of the initial (unknown) phase and intensity (known or constrained) of the light before propagation [1, 6–10]. Retrieving the phase involves finding the phase that best reproduces the known intensities before and after propagation.

1.A. Transport of intensity for coherent light

We will restrict ourselves to scalar (*i.e.* polarization effects are neglected), monochromatic optical fields. Furthermore, for beam–like fields, propagation will primarily be along one direction, which we will denote as the optical axis. Position along the optical axis will be denoted z. A complex field representation U may be employed to model such fields

$$U(x, z) = I\sqrt{(\mathbf{x}, \mathrm{z})}\,\exp[i\phi(\mathbf{x}, z)], \tag{1}$$

where I is intensity, ϕ is phase and $\mathbf{x}$ is position in a plane transverse to the optical axis. A factor of $\exp(ikz)$ has been omitted for simplicity (this represents an identical phase accrued upon propagation for all fields that has no impact upon the measured intensity after propagation and can therefore be neglected.) The propagation of such fields along the optical axis satisfies a paraxial wave equation

$$\frac{\partial}{\partial z} U(\mathbf{x}, z) = \frac{i}{2k}\nabla_{\mathrm{x}}^{2} U(\mathbf{x}, z), \tag{2}$$

where $k = 2\pi/\lambda$ is the wavenumber, where λ is the wavelength of the light and ∇_{x} denotes the Laplacian taken only on the plane transverse to the optical axis. The measured intensity is given by

$$I(\mathbf{x}, z) = |U(\mathbf{x}, z)|^2. \tag{3}$$

Taken together, Eqs (2) & (3) represent free space propagation and may be simplified by taking the z derivative of Eq (3) and employing Eq (2) to replace these axial derivatives of U and U^* with transverse derivatives. After some simplification, one obtains

$$\frac{\partial}{\partial z} I(\mathbf{x}, z) = -\nabla_{\mathrm{x}} \cdot \mathbf{F}(\mathbf{x}, z), \tag{4}$$

where

$$\mathbf{F}(\mathbf{x}, z) = -\frac{i}{2k}\left[U^*(\mathbf{x}, z)\nabla_{\mathrm{x}} U(\mathbf{x}, z) - U(\mathbf{x}, z)\nabla_{\mathrm{x}} U^*(\mathbf{x}, z)\right]. \tag{5}$$

Equation (4) has the form of a continuity equation for intensity. It ascribes the change in intensity at a transverse point x under propagation along the z axis to the flow of intensity into or out of that point. This flow is characterized by a transverse flux F of intensity, defined in Eq (5) [11]. Equation (5) defines the initial flux direction and Eq (4) describes propagation of intensity due to that initial flux direction. These equations are equivalent to a geometrical optics approach in which intensity is propagated along rectilinear rays. To incorporate wave effects, the flux direction must be allowed to change upon propagation, which requires an additional equation.

The propagation of flux is described by the considerably more complex, nonlinear equation

$$\frac{\partial}{\partial z}\boldsymbol{F}(\mathbf{x}, z) = \frac{F^2(\mathbf{x}, z)\nabla_{\mathrm{x}} I(\mathbf{x}, \mathrm{z})}{I^2(\mathbf{x}, z)} - \frac{\nabla_{\mathrm{x}} F^2(\mathbf{x}, z) + 2\mathbf{F}(\mathbf{x}, z)\nabla_{\mathrm{x}} \cdot \mathbf{F}(\mathbf{x}, \mathrm{z})}{2I(\mathbf{x}, z)}$$

$$+\frac{I(\mathbf{x},z)}{2k^2}\nabla_x\left[\frac{\nabla_X^2 I(\mathbf{x},z)}{2I(\mathbf{x},z)}-\frac{|\nabla_x I(\mathbf{x},z)|^2}{4I^2(\mathbf{x},z)}\right], \tag{6}$$

where $F^2 = \mathbf{F}{\cdot}\mathbf{F}$. Conceptually, Eqs. (4) and (6) represent the field through propagation of a flux vector which in turn describes the flow of intensity. The advantage of these equations is that they describe the propagation of real–valued, intuitive quantities. However, these also represent a pair of coupled, nonlinear differential equations which are difficult to solve in practice. Equation (2) describes the same propagation physics as Eqs. (4) and (5) with a linear equation at the cost of adopting a less intuitive, complex-valued representation of the field. Due to the complexity of solving nonlinear equations, Eq (2) is more commonly used to model propagation than Eqs (4) and (5).

For coherent fields, the flux is proportional to the gradient of the field's phase, as can be shown by substituting Eq (1) into Eq (5), yielding

$$\mathbf{F}(\mathbf{x},z) = I(\mathbf{x},z)\,\frac{\nabla_x\phi(\mathbf{x},z)}{k}, \tag{7}$$

which, when inserted into the continuity equation, Eq (4) produces the transport of intensity equation (TIE) [12]

$$\frac{\partial}{\partial z}I(\mathbf{x},z) = -\frac{1}{k}\nabla_x\cdot[I(\mathbf{x},z)\nabla_x\,\phi(\mathbf{x},z)]\ . \tag{8}$$

Equation (8) is a second order, elliptic, partial differential equation that relates the phase ϕ of a coherent field to the intensity I and its axial derivative [13]. Since our interest is in phase retrieval, we will neglect Eq (6) in what follows and use Eq (8) to recover phase from intensity measurements. In practice, the axial derivative of intensity must be approximated by measuring the intensity in two planes that are closely spaced along the optical axis. In that case, the measurement planes must be close enough together so that Eq (6) may be neglected, *i.e.* the flux lines remain approximately straight lines between the two measurement planes.

The TIE is typically solved using Fourier–based methods because of their speed and simplicity. In such solvers, an auxiliary function Θ is introduced satisfying $\nabla_x\Theta = I\,\nabla_x\phi$ which produces from Eq (8) a Poisson's equation of the form

$$g(\mathbf{x},z) = \nabla_X^2\,\Theta(\mathbf{x}), \tag{9}$$

where

$$g(\mathbf{x},z) \equiv -k\frac{\partial}{\partial z}I(\mathbf{x},z) \approx -k\,\frac{I(\mathbf{x},z+\Delta z)-I(\mathbf{x},z-\Delta z)}{2\Delta z} \tag{10}$$

is a finite difference estimate of $\partial I/\partial z$ using measured intensities. Once the auxiliary function is found by solving Eq (9), the phase may be found by solving a second Poisson's equation

$$\nabla_X^2\,\phi(\mathbf{x},z) = \nabla x\cdot\frac{\nabla x\ \Theta(\mathbf{x})}{I(\mathbf{x},z)}, \tag{11}$$

Note that while the use of the auxiliary function can cause some artifacts in the reconstruction when $\nabla_x I$ and $\nabla_x\phi$ are not parallel [14-16], this is often not a problem in practice and does not significantly change the conclusions drawn here. If necessary, a finite difference or finite element–based method could be used to directly solve Eq (8) for phase in order to avoid artifacts due to the auxiliary function.

To simplify the expressions that follow, we will adopt the notation

$$\tilde{f}(\mathbf{u}) = \mathcal{F}[f(\mathbf{x})] = \iint_{-\infty}^{\infty} f(\mathbf{x})\exp(-\mathrm{i}2\pi \mathrm{x}\cdot\mathbf{u})\,\mathrm{d}^2\mathbf{x} \tag{12a}$$

$$f(\mathbf{x}) = \mathcal{F}^{-1}[\tilde{f}(\mathbf{u})] = \iint_{-\infty}^{\infty}\tilde{f}(\mathbf{u})\exp(\mathrm{i}2\pi \mathrm{x}\cdot\mathbf{u})\,\mathrm{d}^2\mathbf{u} \tag{12b}$$

for the Fourier and inverse Fourier transform, respectively and **u** is spatial frequency conjugate to **x**. Equation (9) can be expressed using Fourier transforms as

$$\tilde{g}(\mathbf{u}, z) = -(2\pi)^2\, u^2\, \tilde{\Theta}\,(\mathbf{u}), \tag{13}$$

so that solving for $\tilde{\Theta}$ involves dividing both sides by $-(2\pi)^2\, u^2$. This presents a problem, however, as $u \rightarrow 0$. Aside from the obvious problem with division by zero, Eq (13) implies that the measured intensity variations upon defocus are small for features in Θ with small values of u (large, slowly varying phase features). This means that noise in the measurement will easily overcome any signal for small values of u. Moveover, when dividing by u to reconstruct the auxiliary function, this noise will be significantly amplified. In order to avoid this, regularization is typically used. The most common method due to its ease of implementation and applicability to a broad class of objects is Tikhonov regularization for which the reconstructed auxiliary function takes the form

$$\Theta(\mathbf{x}) = -\, \mathcal{F}^{-1}\left[\frac{\tilde{g}(\mathbf{u})}{H_{\mathrm{LT}}(\mathbf{u})}\right] \tag{14}$$

where

$$H_{\mathrm{LT}}(\mathbf{u}) = \frac{(2\pi)^2\,(u^4 + \rho^4)}{u^2} \tag{15}$$

and ρ is a regularization parameter representing roughly the spatial frequency cutoff below which components of Θ are not reconstructed. Once Θ has been retrieved, the second Poisson's equation for phase, Eq. (11), can be solved using the same method.

2 Transport of intensity for partially coherent light

2.A. The general case

Solving the TIE for coherent fields as described in the preceding section is analogous using measurements of changes in intensity upon propagation to solve Eq. (4) for the flux **F** and then using Eq. (7) to recover phase from the flux. Since the continuity equation expresses energy conservation, Eq (4) is still valid for paraxial fields of any state of coherence. However, due to the random fluctuations in the field, the field cannot be treated as a complex quantity that satisfies the paraxial wave equation. In fact, the idea of phase as a local quantity must be given up. However, in many cases of interest, while the illumination is partially coherent, it is being used to interrogate a deterministic sample, and the goal is to determine the thickness of the sample (or the phase that would be imparted to a coherent field) using a short–defocus method similar to the TIE. Let us first consider the partially coherent illumination. Assume it is a statistically stationary, partially coherent, quasimonochromatic, paraxial, scalar field that can be described in terms of second–order correlations over planes transverse to the optical axis [17]. Fortunately, this model is satisfied by the field emitted by many typical sources (filtered thermal sources or monochromatic LEDs, for example). The function describing the correlation of light between two transverse spatial points $\mathbf{x}_1$ and $\mathbf{x}_2$ in a plane z is the cross–spectral density (CSD) $W(\mathbf{x}_1, \mathbf{x}_2, z)$. The CSD may be represented as an average over an ensemble $\{U(\mathbf{x}, z)\}$ of realizatations of the field that characterize its statistical fluctuations, $W(\mathbf{x}_1, \mathbf{x}_2, z) = \langle U^*(\mathbf{x}_1, z)$ $U(\mathbf{x}_2, z)\rangle$, where $\langle\cdot\rangle$ denotes the ensemble average. Each member of the ensemble satisfies a paraxial wave equation, and so the cross–spectral density satisfies a pair of paraxial wave equations in both $\mathbf{x}_1$ and $\mathbf{x}_2$

$$\frac{\partial}{\partial z} W(\mathbf{x}_1, \mathbf{x}_2, z) = -\frac{i}{2k} \nabla^2_{\mathbf{x}_1} W(\mathbf{x}_1, \mathbf{x}_2, z), \tag{16a}$$

$$\frac{\partial}{\partial z} W(\mathbf{x}_1, \mathbf{x}_2, z) = \frac{i}{2k} \nabla^2_{\mathbf{x}_2} W(\mathbf{x}_1, \mathbf{x}_2, z), \tag{16b}$$

The analogue of intensity for such a field is the spectral density, which is obtained from the cross–spectral density when the points $\mathbf{x}_1$ and $\mathbf{x}_2$ coincide: $S(\mathbf{x}, z) = W(\mathbf{x}, \mathbf{x}, z)$. To derive a form of the TIE with S taking the role of intensity, it is helpful to change variables from $\mathbf{x}_1$ and $\mathbf{x}_2$ to their average and difference $\mathbf{x}' = \mathbf{x}_2 - \mathbf{x}_1$, $\overline{\mathbf{x}} = (\mathbf{x}_1 + \mathbf{x}_2)/2$. Under this change of variables, the sum of the two Eqs (16) is

$$\frac{\partial}{\partial z} W\left(\mathbf{x} - \frac{\mathbf{x}'}{2}, \mathbf{x} + \frac{\mathbf{x}'}{2}, z\right) = -\frac{i}{k} \nabla_{\mathrm{x}} \cdot \nabla_{\mathrm{x}'} W\left(\mathbf{x} - \frac{\mathbf{x}'}{2}, \mathbf{x} + \frac{\mathbf{x}'}{2}, z\right), \tag{17}$$

which, upon taking the limit $\mathbf{x}' \rightarrow \mathbf{0}$, yields

$$\frac{\partial}{\partial z} S(\mathbf{x}, z) = -\nabla_{\mathrm{x}} \cdot \mathbf{F}(\mathbf{x}, z), \tag{18}$$

with

$$\frac{\partial}{\partial z} W\left(\mathbf{x} - \frac{\mathbf{x}'}{2}, \mathbf{x} + \frac{\mathbf{x}'}{2}, z\right) = -\frac{i}{k} \nabla_{\mathrm{x}} \cdot \nabla_{\mathrm{x}'} W\left(\mathbf{x} - \frac{\mathbf{x}'}{2}, \mathbf{x} + \frac{\mathbf{x}'}{2}, z\right), \tag{17}$$

$$\mathbf{F}(\mathbf{x}, z) = -\frac{i}{k} \nabla_{\mathrm{x}'} W\left(\mathbf{x} - \frac{\mathbf{x}'}{2}, \mathbf{x} + \frac{\mathbf{x}'}{2}, z\right)\Big|_{\mathrm{x}' = 0} \tag{18}$$

Notice that although this is a generalization of the TIE to partially coherent light, it differs from the TIE in an important way. For a coherent field, knowing I and ϕ is sufficient to fully describe the field as well as predict its intensity over all space by solving Eq (2). For partially coherent fields, W is required to fully describe the field and its propagation. However, solving Eqs (18) and (19) only provide information about local variations in the cross–spectral density. Recovering W from intensity measurements will generally require many more measurements of intensity at different propagation distances [18-21].

Next, consider partially coherent illumination striking a deterministic sample [22, 23]. For simplicity, we will consider a thin sample illuminated by a partially coherent beam described by cross–spectral density $W_{\mathrm{i}}(\mathbf{x}_1, \mathbf{x}_2, z)$. The sample can be described by a transmission function $\sqrt{T(\mathbf{x})}\,\exp[ikL(\mathbf{x})]$, where T is the intensity transmission of the sample and L is the total optical path length through the sample at a point $\mathbf{x}$ such that kL would represent the phase imparted to coherent light passing through the sample. The partially coherent field after the sample is given by

$$W_{\mathrm{o}}(\mathbf{x}_1, \mathbf{x}_2, z) = \sqrt{T(\mathbf{x}_1)T(\mathbf{x}_2)}\, \exp\{ik[L(\mathbf{x}_2) - L(\mathbf{x}_1)]\}\, W_{\mathrm{i}}(\mathbf{x}_1, \mathbf{x}_2, z). \tag{20}$$

The flux of this output field is then given by

$$\mathbf{F}_{\mathrm{o}}(\mathbf{x}, z) = T(\mathbf{x}) S_{\mathrm{i}}(\mathbf{x}, z)\, \nabla L(\mathbf{x}) + T(\mathbf{x}) \mathbf{F}_{\mathrm{i}}(\mathbf{x}, z), \tag{21}$$

where $\mathbf{F}_{\mathrm{i}}(\mathbf{x}, z)$ is the flux and $S_{\mathrm{i}}(\mathbf{x}, z)$ the spectral density of the incident illumination. The first term of the output flux $\mathbf{F}_{\mathrm{o}}$ has the form of the flux of a coherent field of intensity $T(\mathbf{x})S_i(\mathbf{x}, z)$ and phase $kL(\mathbf{x})$. The second term behaves as the incident flux attenuated by passage through the sample, but otherwise unperturbed. If $S_{\mathrm{o}}(\mathbf{x}, z)$ represents the spectral density measured after the sample, it must satisfy the equation

$$\frac{\partial}{\partial z} S_{\mathrm{o}}(\mathbf{x}, z) = -\nabla_{\mathrm{x}} \cdot \mathbf{F}_{\mathrm{o}}(\mathbf{x}, z) = -\nabla_{\mathrm{x}} \cdot [T(\mathbf{x}) S_{\mathrm{i}}(\mathbf{x}, z) \nabla L(\mathbf{x})] - T(\mathbf{x}) \nabla_{\mathrm{x}} \cdot \mathbf{F}_{\mathrm{i}}(\mathbf{x}, z) - [\nabla_{\mathrm{x}} T(\mathbf{x})] \cdot \mathbf{F}_{\mathrm{i}}(\mathbf{x}, z). \tag{22}$$

This suggests a measurement scheme to remove the effects of illumination. By focusing the detector on the sample and then removing the sample, an image of S_i can be recorded. Then by performing a TIE–type measurement of the illumination, one could obtain

$$\nabla_{\mathrm{x}} \cdot \mathbf{F}_{\mathrm{i}}(\mathbf{x}, z) = -\frac{\partial}{\partial z} S_i(\mathbf{x}, z) \tag{23}$$

which can be used to replace the term $\nabla_x \cdot \mathbf{F}_i$ in Eq (22). Furthermore, solving this equation for $\mathbf{F}_i$ will allow its replacement in the final term of Eq (22). Next, an in-focus measurement taken with the sample in place yields $T(\mathbf{x})S_i(\mathbf{x}, z)$. By comparing the in–focus images with and without the sample in place, $T(\mathbf{x})$ can be calculated. Now, Eq (22) can be rearranged as

$$\frac{\partial}{\partial z} S_o(\mathbf{x}, z) - T(\mathbf{x}) \frac{\partial}{\partial z} S_i(\mathbf{x}, z) + \nabla_x T(\mathbf{x}) \cdot \mathbf{F}_i(\mathbf{x}, z) = -\nabla_x \cdot [T(\mathbf{x}) S_i(\mathbf{x}, z) \nabla L(\mathbf{x})] \tag{24}$$

where the right–hand side has a form similar to the TIE. The left–hand side is known or can be calculated from measurements of the illumination and sample intensities.

Equation (24) represents a method to remove partially coherent illumination effects from TIE measurements in order to characterize sample thickness. However, we have not addressed the defocus distance used when measuring $\partial S_i/\partial z$ and $\partial S_0/\partial z$. In practice, this must be evaluated by measuring the spectral density in two closely spaced planes. The spacing between planes must be kept small enough so that higher–order diffraction effects do not become significant. For coherent plane–wave illumination, this requirement depends on the maximum spatial frequency present in the sample's transmission function, u_{max} through $z_{max} << 1/(\pi\lambda u^2_{max})$. For partially coherent illumination, an additional requirement is that the illumination's cross–spectral density is slowly–varying so that $W(\mathbf{x} - \lambda z \mathbf{u}_{max}, \mathbf{x} + \lambda z \mathbf{u}_{max}, z)$ may be well–approximated by a first–order Taylor expansion in $\lambda z \mathbf{u}_{max}$.

The validity of Eq (24) was demonstrated in [23]. Consider the following situation. Partially coherent illumination is produced by a circular source placed in the back focal plane of a collimating lens. The resulting illumination is uniform in intensity but partially coherent. An illumination modulation mask is placed in the path of this beam to render the illumination non–uniform. It consists of five vertical bars of OPL 0.3 µm. The sample is placed a further 200 µm from the illumination modulation mask. Results were simulated using numerical Fresnel propagation with additive Gaussian noise (to simulate thermal noise). Results for a TIE–type measurement of this sample are illustrated in Fig 1.

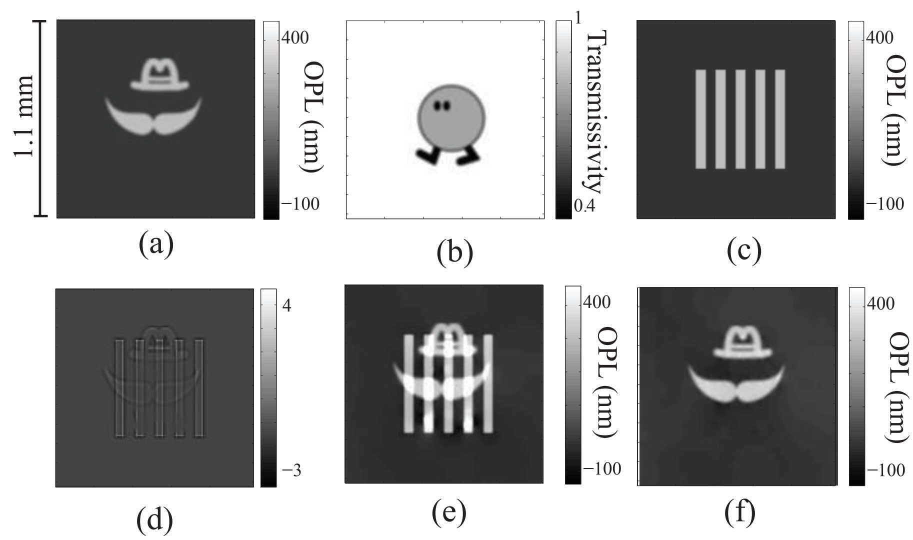

Fig 1. Simulation results for partially coherent TIE. (a) Sample's OPL and (b) transmittance. (c) The OPL of the modulation mask. (d) The difference in defocused intensity measurements with both masks in place. (e) Results of OPL reconstruction assuming coherent, plane wave illumination, Eq. (8), with both sample and modulation mask in place. (f) The result of applying Eq (24).

2.B. Homogeneous, Schell-model source or Köhler illumination

Equation (24) describes a method to recover the thickness of a deterministic sample under partially coherent illumination provided the defocus distance is short enough. However, in many cases the main effect of partial coherence is to produce illumination that is essentially a collection of mutually incoherent plane waves. In these cases, the effect of illumination over larger defocus distances is easy to model from geometrical considerations.

Consider a fully incoherent primary source placed at the front focal plane of a lens of focal length f. A point on the source located at transverse position $\mathbf{x}$ will produce a plane wave propagating along a vector given by $\hat{z} - \mathbf{x}/f$ as illustrated in Fig 2(a). For a fully incoherent primary source, no plane wave will interfere with any other, and so the intensities carried by each plane wave add at the detector. Now, consider a sample placed in the path of this collimated beam. Information from any point on the sample is carried by the collection of plane waves making up the beam, and therefore leaves the sample at a range of angles. If the source is a disk of diameter D, at a defocus distance Δz, the information from point on the sample blurs to a disk of diameter $\Delta z D/f$ as illustrated in Fig 2(b). More generally, the intensity that would be produced in a plane Δz after the sample is blurred by convolution with the source's intensity distribution scaled by a factor $-\Delta z/f$.

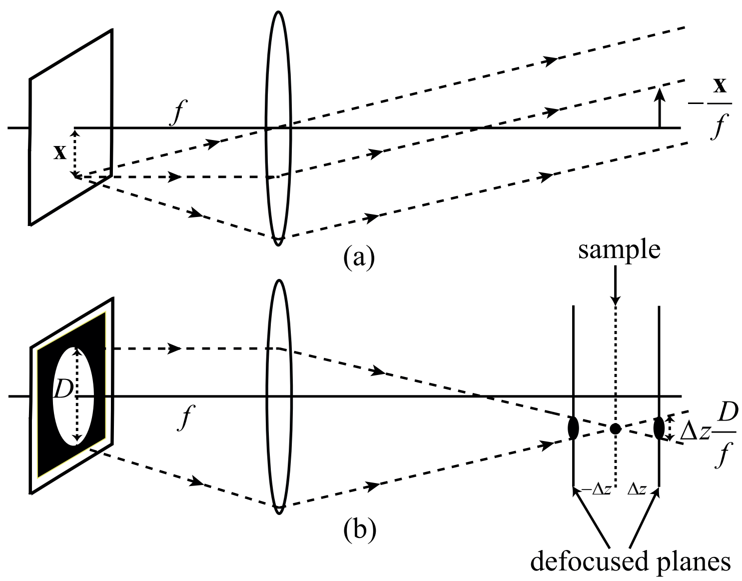

Fig 2. The effect of Köhler illumination. (a) A single off-axis point located at transverse position $\mathbf{x}$ in the source plane will produce a collimated beam whose tilt is characterized by the transverse direction vector $-\mathrm{x}/f$. (b) A source centered on the optical axis of diameter D will produce a set of plane waves whose tilts will cause a point on the sample plane to defocus to a blur spot of size $\Delta z\mathrm{D}/f$ in the defocused planes $\pm\Delta z$.

In the case of TIE, the defocused intensity for a coherent, normally incident plane wave can be found from Eq (8),

$$I(\mathbf{x}, z+\Delta z) = I(\mathbf{x}, z) - \frac{\Delta z}{k} \nabla_{\mathrm{x}} \cdot [I(\mathbf{x}, z) \nabla_{\mathrm{x}} \phi(\mathbf{x}, z)] \tag{25}$$

The net effect for an extended, incoherent, primary source of spectral density $S_{\mathrm{i}}(\mathbf{x})$ in the source plane is

$$\frac{\partial S_{\mathrm{o}}(\mathbf{x}, z)}{\partial z} \approx -k \frac{S_{\mathrm{o}}(\mathbf{x}, z+\Delta z) - S_{\mathrm{o}}(\mathbf{x}, z-\Delta z)}{2\Delta z} = S_{\mathrm{i}}\left(-\frac{f\mathbf{x}}{\Delta z}\right) * \{\nabla_{\mathrm{x}} \cdot [I(\mathbf{x}, z) \nabla_{\mathrm{x}} \phi(\mathbf{x}, z)]\} \tag{26}$$

The term in braces on the right hand side is the usual coherent TIE measurement. The use of an incoherent primary source produces a blurred TIE measurement. This can be modeled as convolution of the coherent TIE measurement of defocused intensity with the source's intensity distribution. This results in a reduction in resolution of the reconstructed phase. To see this, consider the introduction of an auxiliary function $\nabla_x \Theta = I \nabla_x \phi$, a common method used to solve the TIE. If the finite difference of measured spectral densities on the left–hand side is denoted as $g(\mathbf{x}, z)$, then Eq (26) becomes

$$g(\mathbf{x}, z) = S_{\mathrm{i}}\left(-\frac{f\mathbf{x}}{\Delta z}\right) * \nabla_{\mathbf{X}}^2 \Theta(\mathrm{x}). \tag{27}$$

If this equation is solved for Θ, then ϕ may be recovered by solving the Poisson's equation

$$\nabla_{\mathbf{X}}^2 \phi\,(\mathbf{x}, z) = \nabla_{\mathrm{x}} \cdot \left[\frac{\nabla_{\mathrm{x}} \Theta(\mathbf{x})}{I(\mathbf{x}, z)}\right] \tag{28}$$

which is typically done using fast Fourier transform–based solvers.

Equation (27) can be solved by using Fourier transforms as well. Its Fourier transform from **x** to **u** yields

$$\tilde{g}(\mathbf{u}, z) = -(2\pi)^2\, u^2\, H_s(\mathbf{u})\, \tilde{\Theta}(\mathbf{u}), \tag{29}$$

where

$$H_S(\mathbf{u}) = \left(\frac{\Delta z}{f}\right)^2 \tilde{S}_i\left(-\frac{\Delta z\mathbf{u}}{f}\right) \tag{30}$$

is a transfer function associated with blurring with the finite source. The Laplacian may be inverted by division of Eq (27) by H_{LT} as in Eq (14) which would lead to a blurred estimate of Θ, *i.e.*

$$S_{\mathrm{i}} = \left(-\frac{f\mathbf{x}}{\Delta z}\right) * \Theta(\mathrm{x}) = -\mathcal{F}^{-1}\left[\frac{\tilde{g}(\mathbf{u}, z)}{H_{\mathrm{LT}}(\mathbf{u})}\right]. \tag{31}$$

Removing this blur involves deconvolution, or division by H_s in Eq (27). This is typically an extremely difficult procedure since H_s often has either nulls or takes on small values at high frequencies which describes poor measurement of higher spatial frequency components of Θ. During reconstruction, noise at these spatial frequencies will tend to be significantly amplified, corrupting the reconstructed phase. This may be somewhat mitigated by regularization if desired. We have found that although Tikhonov regularization can mitigate both the high– and low–frequency artifacts in this case, the regularization parameters required are significantly different so that acceptable results require two–step regularization, *i.e*. both the inverse Laplacian and deconvolution must be regularized independently.

$$\Theta(\mathrm{x}) = -\mathcal{F}^{-1}\left[\frac{\tilde{g}(\mathbf{u}, z)}{H_{\mathrm{ST}}(\mathbf{u})\, H_{\mathrm{LT}}(\mathbf{u})}\right], \tag{32}$$

where the Tikhonov regularized deconvolution transfer function is given by

$$H_{\mathrm{ST}}(\mathbf{u}) = \frac{|H_{\mathrm{S}}(\mathbf{u})|^2 + \rho_s}{\mathrm{H}_{\mathrm{S}}^*(\mathbf{u})}, \tag{33}$$

In principle, for a system with Köhler illumination and unlimited exposure time, using a pinhole as a primary source would alleviate this blurring. In practice, using a smaller source means fewer photons which requires a longer integration time to produce acceptable images. There is, therefore, a tradeoff between exposure time and source size. The results of this tradeoff are illustrated for simulated measurement in Fig 3. In this simulation, the target was assumed to be a star etched 50 nm deep in glass of refractive index 1.5. Pixel size was taken to be 2.2 μm. Defocus distance used in the TIE measurement scheme was Δz = 500 μm.

Thermal noise was simulated using additive Gaussian noise with fixed variance (to simulate fixed exposure time). Circular sources of radii 1.5, 2.7 and 3.5 mm were simulated using a collimating lens of focal length f = 75 mm. Results for this simulation are illustrated in Fig 3. Using larger sources improves the measured signal to noise ratio since the detector captures more signal photons during the exposure time. However, a larger source also produces more blur or more deconvolution artifacts.

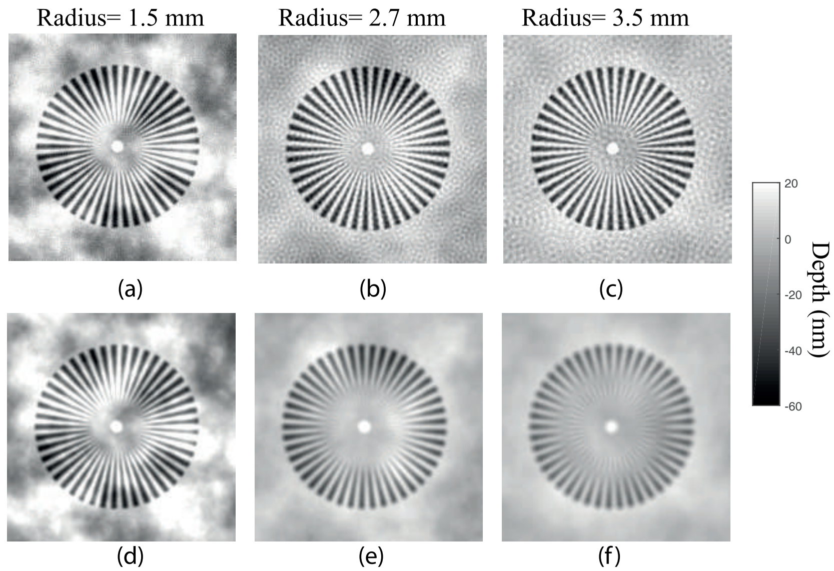

Fig 3. The effect of primary source size in a Köhler illumination system on resolution using a star target. (a)-(c) Reconstructed phase with deconvolution using Eq (32) with moderate Tikhonov regularization ρ_s = 1000 (a.u.) for indicated source radii. (d)-(f) Reconstructed, blurred phase using Eq (31) for indicated source sizes.

3 Conclusion

The TIE is one method of phase retrieval that uses the physics of free space propagation to enable the recovery of phase from intensity measurements under short propagation distances. For both coherent and partially coherent light, the TIE is equivalent to a continuity equation relating intensity changes upon propagation to a flux vector describing the flow of intensity, which for stationary fields is equivalent to energy conservation. The TIE enables the recovery of flux from intensity measurements.

For coherent fields, this flux is proportional to the gradient of the field's phase and therefore, the retrieved flux can be used to retrieve phase. For partially coherent fields, there is no concept of local phase that is equivalent to that in a coherent field. However, since the problem is usually the retrieval of sample thickness or depth, one can recast the problem as obtaining the sample's thickness from its impact upon the total flux of the light leaving the sample. This is found to depend on the flux and intensity of the illuminating partially coherent field as well as the attenuation and optical path length of the sample. Finally, it was demonstrated that for the case of Köhler illumination, a major effect of the illumination is blurring of the resulting phase. To some extent this can be compensated for by deconvolution methods.

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