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Statistics of derivatives of intensity and phase in fractal speckles

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This article is dedicated to Prof T Asakura

Statistical properties of the derivatives of the intensity and phase in fractal speckles are investigated theoretically. To obtain the statistics, we derive two key parameters in the joint density function of speckle intensity, phase and their derivatives with respect to x and y. These parameters in fractal speckles are given by performing the integration of a negative power function, which corresponds to an intensity distribution incident on a diffuser for producing fractal speckles. In relation to these two parameters, we also derive a correlation area in fractal speckles. The results show that the two parameters and the correlation area in fractal speckles obey power functions related to the negative power exponent in the function of the intensity profile incident on the diffuser.

Keywords: Fractal, Speckle, Statistics of derivatives, Power function

1 Introduction

It has been known that speckle patterns with fractal properties are produced when coherent light with an intensity distribution obeying a negative power function is incident on a diffuser such as a ground glass plate [1]. Such speckles have an extremely long spatial correlation tail decaying with a negative power law related to the intensity profile incident on the diffuser. Since such a correlation function is one of the major characteristics of fractals, unconventional speckles of this type may be called fractal speckles [2]. The correlation properties of fractal speckles have been studied theoretically and experimentally as well as by computer simulations in three optical regions; Fraunhofer and Fresnel diffraction regions and an image plane of a diffuser [3]. The results showed that intensity distributions in these regions share the same fractal properties, which correspond to long power-law tails in their spatial correlation functions. This feature is supposed to extend measurement ranges in various metrological applications of speckle patterns based on their correlation properties [4]. Another application of fractal speckles is an optical formation of fractal random media in view of a random laser, in which three-dimensional fractal speckle fields are generated by two or three speckle beams crossed orthogonally in computer simulations and showed that their intensity distributions exhibit the spatial correlation functions obeying a negative power law [5]. From the view of the interesting physical properties and practical applications in fractal speckles, their statistical properties and fractal structures are studied in details, such as multifractality [6] and lacunarity [7] of intensity in fractal speckles, and the correlation properties of clipped fractal speckle intensities [8]. The phase statistics of fractal speckles in the Fraunhofer region of a diffuser are also discussed by computer simulations and shown to have another kind of fractality [9].

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In applications of statistical properties of speckles, the statistics of the derivatives of speckle intensity and phase play an important role in some problems, such as the properties of local maxima of speckle intensity and the level crossing problems related to the zeros of the derivative of speckle intensity, the statistical properties of the geometrical ray directions in speckles, which is given by the local gradient of phase related to the derivative of speckle phase, the density of vortex and so on [10]. While there are physical and practical research backgrounds, the statistical properties of the derivatives of the intensity and phase in fractal speckles have not been studied yet. In the present paper, therefore, we investigate the theoretical derivation of statistics of derivatives of intensity and phase in fractal speckles.

2 Statistics of derivatives of intensity and phase in speckle patterns

For fully developed speckle, when it is assumed that the intensity distribution incident on a diffuser has axial symmetry, a probability density function of the derivatives of speckle intensity and phase is derived from a joint probability density function of six random variables of intensity and phase in the speckle field. The joint probability density function is expressed as [10]

$$p(I, \theta, I_x, \theta_x, I_y, \theta_y) = \frac{1}{64\pi^3\sigma^2 b_x b_y} \exp\left[-\frac{4b_x b_y I^2 + \sigma^2 (b_y I_x^2 + b_x I_y^2) + 4\sigma^2 I^2 (b_y \theta_x^2 + b_x \theta_y^2)}{8I\sigma^2 b_x b_y}\right], \tag{1}$$

where I and θ are the intensity and phase in a speckle field, I_x and θ_x are the partial derivatives of I and θ with respect to x, and I_y and θ_y are the partial derivatives of I and θ with respect to y. σ^2 is an autocorrelation function of the real and imaginary parts in complex amplitude of speckle fields. b_x and b_y are autocorrelation functions of partial derivatives with respect to x and y of real and imaginary parts in speckle fields. The three parameters σ^2, b_x and b_y are the eigen values of the covariance matrix obtained from the process of the derivation of the joint probability density function in Eq (1) and are given by

$$\sigma^2 = \frac{\kappa}{2\lambda^2 z^2} \iint I(x,y)dxdy, \tag{2}$$

$$b_x = \frac{2\kappa\pi^2}{\lambda^4 z^4} \iint x^2\, I(x,y)dxdy, \tag{3}$$

$$b_y = \frac{2\kappa\pi^2}{\lambda^4 z^4} \iint y^2\, I(x,y)dxdy, \tag{4}$$

where κ is a proportionality constant related to the correlation area of the diffuser, λ is a wavelength of an optical source, z is a propagation distance of the speckle fields and $I(x,y)$ is the intensity profile incident on the diffuser in the x-y coordinate. The integral regions span from $-\infty$ to ∞, which is omitted for brevity in the above equations and in the rest of the paper unless specified otherwise. Since the probability density functions of the derivatives of speckle intensity and phase are derived as marginal densities for each random variable, the statistical properties of the derivatives are governed by these three parameters.

3 Theoretical background of fractal speckles

Speckle patterns with fractal property can be generated by illuminating a diffuser with an intensity distribution obeying a negative power-law function expressed as [1, 2]

$$I(r) = r^{-D}, \tag{5}$$

where r is the radial coordinate. In early studies of fractal speckles, the theoretical derivation and the experimental setup are based on doubly scattered speckle, in which the intensity distribution in Eq (5) is produced by the Fraunhofer diffraction pattern of the scattered wave from a random fractal object with fractal dimension D and then is incident on the diffuser. However, due to the development of digital optical devices, the intensity distribution in Eq (5) is directly produced by a computer generated hologram using a spatial light modulator [11, 12]. For the generation of fractal speckles by this method, the theoretical background of speckles on the basis of a single scattering speckle is suitable [13]. The amplitude correlation coefficient of speckle patterns is given by the Fourier transform of the intensity distributions of the scattering spot in Eq (5). Since the negative power-law function in Eq (5) has circular symmetry, the Fourier transform can be converted into the Fourier-Bessel transform using the transformation of variables from the Cartesian coordinate to the polar coordinate [14]. After performing the transformation, we have the amplitude correlation coefficient

$$\mu_A(\Delta\rho) = \frac{1}{2\pi}\int_0^{\infty} r^{1-D}\ J_0\left(\frac{2\pi}{\lambda z}\Delta\rho r\right) dr, \tag{6}$$

where $J_\nu(\cdot)$ is the νth order Bessel function of the first kind and $\Delta\rho$ is the difference of the radial coordinate. Eq (6) reduces to [15],

$$\mu_A(\Delta\rho) = \frac{2^{-(D+1)}}{\pi}\ \frac{\Gamma\left(\dfrac{2-D}{2}\right)}{\Gamma\left(\dfrac{D}{2}\right)}\left(\frac{2\pi}{\lambda z}\Delta\rho\right)^{-(2-D)} \tag{7}$$

for $1/2 < D < 2$, where $\Gamma(\cdot)$ is the gamma function. The intensity correlation coefficient of speckle patterns is given by the square modulus of the amplitude correlation coefficient, and therefore, we have

$$\mu_I(\Delta\rho) = |\mu_A(\Delta\rho)|^2 \propto \Delta\rho^{-2(2-D)} \qquad \text{for } 1/2 < D < 2. \tag{8}$$

Fractality of the intensity distribution of fractal speckles is evaluated by fractal dimension. In this case, fractal dimension D_s of fractal speckles is calculated on the basis of the concept of mass fractals and is given by

$$D_s = 2D - 2. \tag{9}$$

Equations (7) and (8) holds for $1/2 < D < 2$. However, Eq (9) gives negative values or zero in the range of $1/2 < D \le 1$, which is to be interpreted as $D_s = 0$ for $D \le 1$, and speckle patterns do not have fractality in this range of D. Therefore, we finally obtain

$$\mu_I(\Delta\rho) \propto \Delta\rho^{-2(2-D)} \qquad \text{for } 1 < D < 2. \tag{10}$$

The results agree well with those obtained on the basis of the doubly scattered speckles [1,2].

4 Statistics of derivatives of intensity and phase in fractal speckles

To derive statistical properties of the derivatives of the intensity and phase in fractal speckles, we use a realistic model of the intensity distributions of scattering spots. It is given by applying the Fisher-Burford approximation and an approximation using a Gaussian function to Eq (5) [8, 16] and is expressed as

$$I(r) = \left[1 + \left(\frac{r}{R}\right)^2\right]^{-D/2} \exp\left(-\frac{r^2}{\alpha^2}\right), \tag{11}$$

where R is a parameter adjusting deviations from Eq (5) around the origin and is regarded to determine the maximum speckle size in fractal speckles, and α is a parameter limiting the extent of scattering spots and is regarded to give the minimum speckle size in fractal speckles. By substituting Eq (11) into Eqs (2)-(4), transforming the Cartesian coordinate to the polar coordinate and performing the integration with respect to the angle θ, Eqs (3) and (4) become the same function $b_r = b_x = b_y$ due to the symmetry of the intensity distribution of the scattering spot, and we have

$$\sigma^2 = \frac{\pi\kappa}{\lambda^2 z^2}\int_0^{\infty}\left[1 + \left(\frac{r}{R}\right)^2\right]^{-D/2} \exp\left(-\frac{r^2}{\alpha^2}\right) r dr, \tag{12}$$

$$b_r = \frac{2\kappa\pi^3}{\lambda^4 z^4} \int_0^\infty r^2 \left[1 + \left(\frac{r}{R}\right)^2\right]^{-D/2} \exp\left(-\frac{r^2}{\alpha^2}\right) \; rdr. \tag{13}$$

It is noticed that Eq (12) has the similar form as the integration in the derivation of the contrast of spatially integrated fractal speckles [7]. After performing the transformation of $r^2 = R^2 r'$, Eq (12) reduces to

$$\sigma^2 = \frac{\pi\kappa R^2}{2\lambda^2 z^2} \int_0^\infty (1 + r')^{-D/2} \exp\left(-\frac{R^2}{\alpha^2} r'\right) \; dr'. \tag{14}$$

By using the integral representation of the Whittaker function [15]

$$W_{k,l}(z') = \frac{z'^k}{\Gamma(l - k + 1/2)} \exp\left(-\frac{z'}{2}\right) \int_0^\infty t^{l-k-(1/2)} \exp(-t) \left(1 + \frac{t}{z'}\right)^{l+k-(1/2)} dt \tag{15}$$

for $\mathrm{Re}(l - k) > -1/2$ and $|\arg z'| < \pi$, we have finally

$$\sigma^2 = \frac{\pi\kappa R^2}{2\lambda^2 z^2} \left(\frac{R^2}{\alpha^2}\right)^{(D-4)/4} \exp\left(\frac{R^2}{2\alpha^2}\right) \; W_{k_1,l_1} \left(\frac{R^2}{\alpha^2}\right), \tag{16}$$

where $k_1 = -D/4$ and $l_1 = (2 - D)/4$.

Next, we calculate Eq (13). Using the transformation of variables similar to Eq (14), we have

$$b_r = \frac{\kappa\pi^3 R^4}{\lambda^4 z^4} \exp\left(\frac{R^2}{\alpha^2}\right) \int_1^\infty r'^{-D/2} (r' - 1) \exp\left(-\frac{R^2}{\alpha^2} r'\right) \; dr'. \tag{17}$$

This equation is also calculated using the integral representation of the Whittaker function in Eq (15) and reduces to

$$b_r = \frac{\kappa\pi^3 R^4}{\lambda^4 z^4} \left(\frac{R^2}{\alpha^2}\right)^{-(6-D)/4} \exp\left(\frac{R^2}{2\alpha^2}\right) \; W_{k_2,l_2} \left(\frac{R^2}{\alpha^2}\right) \; , \tag{18}$$

where $k_2 = -(D + 2)/4$ and $l_2 = (D - 4)/4$.

An important parameter related with these parameters σ^2 and b_r is a correlation area. Rigorously speaking, the correlation area cannot be defined in concept of fractals because fractal structures do not have a characteristic length and therefore the correlation area diverges. However, in this study, since the intensity distribution of the scattering spots is defined as Eq (11) and has finite extents in respect to R and α, the correlation area can be calculated. The correlation area of fractal speckles is expressed as

$$A_c = (\lambda z)^2 \frac{\iint I(x,y)^2 \; dxdy}{[\iint I(x,y) dxdy]^2} \; . \tag{19}$$

While the denominator in Eq (19) is given by Eq (16), the numerator is obtained from

$$I_n = 2\pi \int_0^\infty \left[1 + \left(\frac{r}{R}\right)^2\right]^{-D} \; \exp\left(-\frac{2r^2}{\alpha^2}\right) dr. \tag{20}$$

This integration is also calculated by the similar fashion as σ^2 and reduces to

$$I_n = \pi R^2 \left(\frac{2R^2}{\alpha^2}\right)^{(D-2)/2} \; \exp\left(\frac{R^2}{\alpha^2}\right) W_{k_3,l_3} \left(\frac{2R^2}{\alpha^2}\right), \tag{21}$$

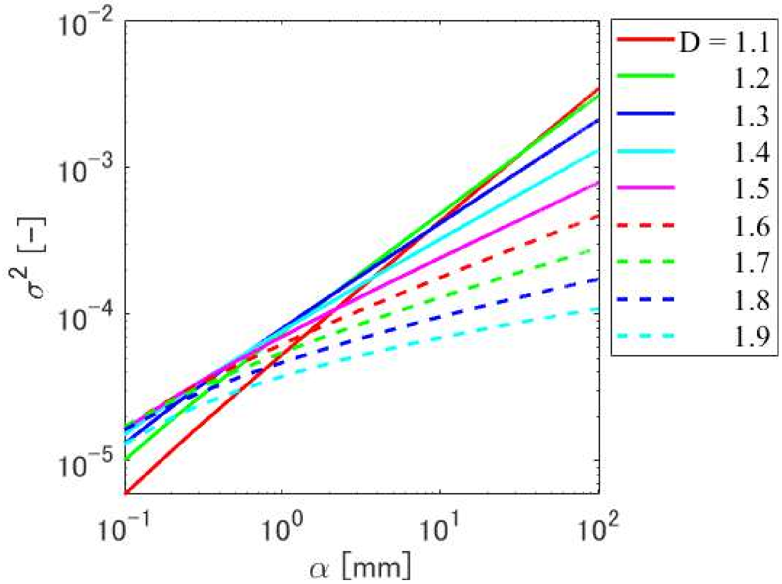
where $k_3 = -D/2$ and $l_3 = (1 - D)/2$. Substituting Eqs (16) and (21) into Eq (19) yields

$$A_c = \frac{\lambda^2 z^2}{\pi R^2} \, 2^{(D-2)/2} \; \left(\frac{R}{\alpha}\right)^2 \frac{W_{k_3,l_3} \, (2R^2/\alpha^2)}{[W_{k_1,l_1} \, (R^2/\alpha^2)]^2}. \tag{22}$$

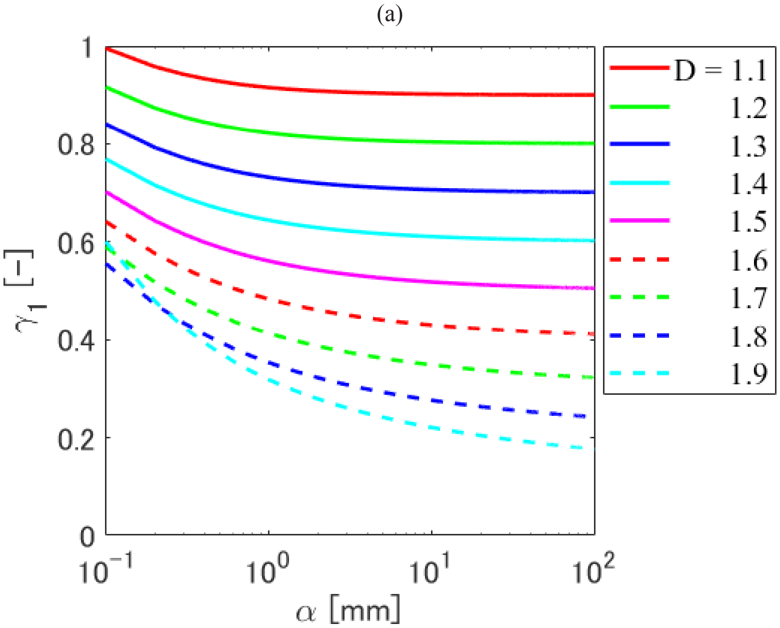
Figures 1, 2, 3 and 4 show the logarithmic plots of Eqs (16), (18), (22) and b_r/σ^2, which is the quantity related to three parameters σ^2, b_r and A_c , and their slopes γ_n, where n stands for a sequential number of these four quantities. These figures are obtained by setting $\kappa = 25$ (μm^2), $R = 10$ (μm), α in the range from 100 (μm) to 10 (m), $\lambda = 632.8$ (nm) and $z = 0.15$ (m). It is seen from these figures that the values of these parameters linearly vary in the logarithmic plot and their slopes converge to a constant with an increase in α. We can derive the asymptotic values of Eqs (16), (18), (22) and b_r/σ^2 for $\alpha \to \infty$ [17]. The Whittaker function in Eq (15) is also represented by [15]

$$W_{k,l}(z') = \frac{\Gamma(-2l)}{\Gamma(1/2 - l - k)} M_{k,l}(z') + \frac{\Gamma(2l)}{\Gamma(1/2 + l - k)} M_{k,-l}(z'), \tag{23}$$

in which $M_{k,l}(z')$ is represented by the confluent hypergeometric function $F(\cdot,\cdot;\cdot)$ as



(a)



(b)

Fig 1.(a) Logarithmic plot of σ^2 as a function of α in the entire range of D, and (b) its local slope γ_1.

$$M_{k,l}(z') = z'^{\,l+1/2} \exp\left(-\frac{z'}{2}\right) F(l - k + 1/2, 2l + 1; z'). \tag{24}$$

Substituting Eqs (23) and (24) into Eqs (16), (18) and (22) and noting that $F(l - k + 1/2, 2l + 1; z') \to 1$ as $\alpha \to \infty$, we have

$$\sigma^2 = \frac{\pi\kappa R^2}{2\lambda^2 z^2}\left[a_1 + b_1\left(\frac{R^2}{\alpha^2}\right)^{-(2-D)/2}\right] = \frac{\pi\kappa R^2}{2\lambda^2 z^2}\Gamma\left(\frac{2-D}{2}\right)\left(\frac{\alpha}{R}\right)^{2-D}, \tag{25}$$

$$b_r = \frac{\kappa\pi^3 R^4}{\lambda^4 z^4}\left[a_2\left(\frac{R^2}{\alpha^2}\right)^{-(4-D)/2} + b_2\right] = \frac{\kappa\pi^3 R^4}{\lambda^4 z^4}\Gamma\left(\frac{D-4}{2}\right)\left(\frac{\alpha}{R}\right)^{4-D}, \tag{26}$$

$$I_n = \pi R^2\left[a_3 + b_3\left(\frac{2R^2}{\alpha^2}\right)^{(D-1)}\right] = \pi R^2, \tag{27}$$

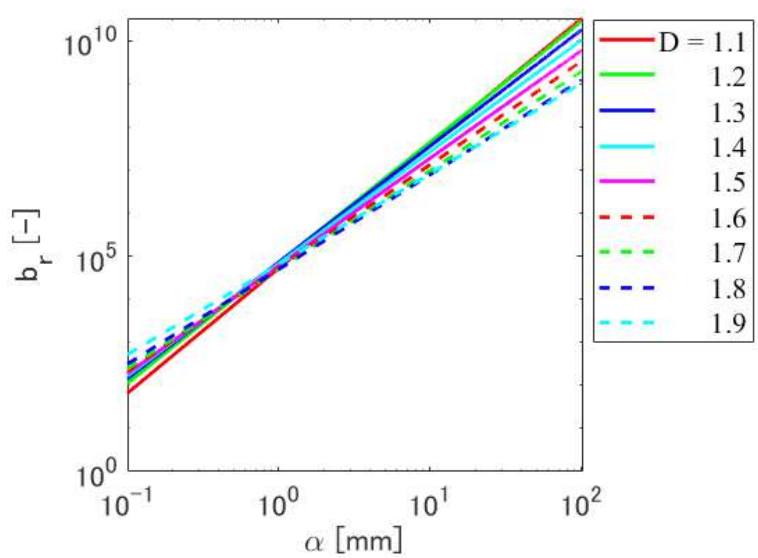
where a_n and b_n are constant coefficients in the right hand side of the Whittaker function in Eq (23), namely $a_n = \Gamma(-2l_n)/\Gamma(1/2 - l_n - k_n)$ and $b_n = \Gamma(2l_n)/\Gamma(1/2 + l_n - k_n)$. Using Eqs (25)-(27), the correlation area and b_r/σ^2 reduce to

$$A_c = \frac{\lambda^2 z^2}{\pi R^2}\Gamma\left(\frac{2-D}{2}\right)^{-2}\left(\frac{\alpha}{R}\right)^{2D-4}, \tag{28}$$

$$\frac{b_r}{\sigma^2} = \frac{2\pi^2 R^2}{\lambda^2 z^2}\Gamma\left(\frac{D-4}{2}\right)\Gamma\left(\frac{2-D}{2}\right)^{-1}\left(\frac{\alpha}{R}\right)^2, \tag{29}$$

$$= \frac{2\pi}{A_c}\Gamma\left(\frac{D-4}{2}\right)\Gamma\left(\frac{2-D}{2}\right)^{-3}\left(\frac{\alpha}{R}\right)^{2D-2}. \tag{30}$$

These approximations agree well with the numerical results evaluated from theoretical analyses shown in the figures.



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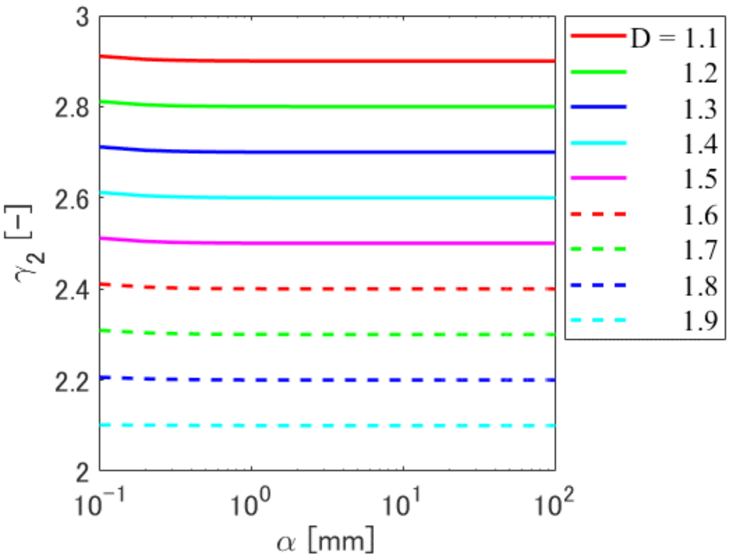
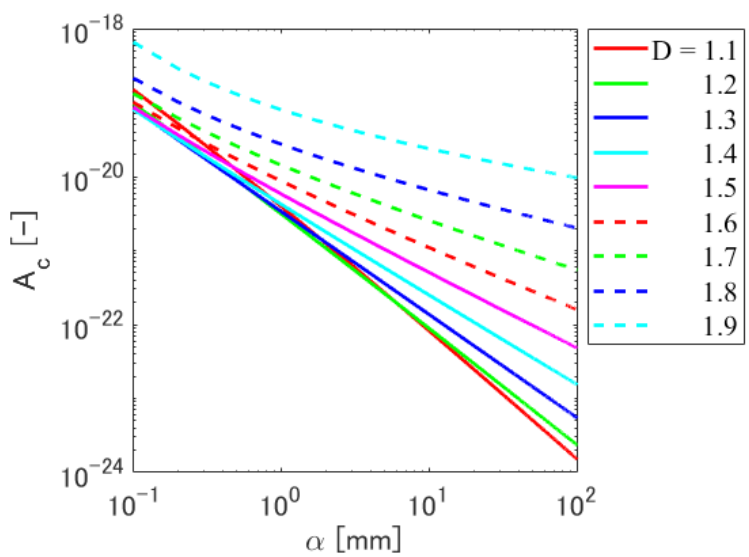


(b)

Fig 2.(a) Logarithmic plot of b_r as a function of α in the entire range of *D,* and (b) its local slope γ_2.

In the case of ordinary speckles generated from a scattering spot with circular symmetry such as the circular and the Gaussian functions, the four parameters obey $\sigma^2 \propto \alpha^2$, $b_r \propto \alpha^4$, $A_c \propto \alpha^{-2}$, and $b_r/\sigma^2 \propto A_c^{-1} \propto \alpha^2$, in which α is regarded as the radius of the circular function or the width of e^{-1} intensity of the Gaussian function. Therefore, it is found in fractal speckles that σ^2 and b_r gradually increase and A_c gradually decreases with an increase in α, with the dependence unlike ordinary speckles, while the power exponent of b_r/σ^2 of fractal speckles converges as α increases to the same dependence with ordinary speckles.



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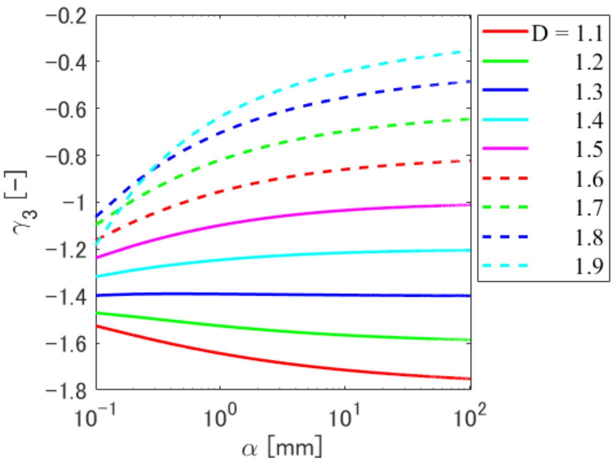
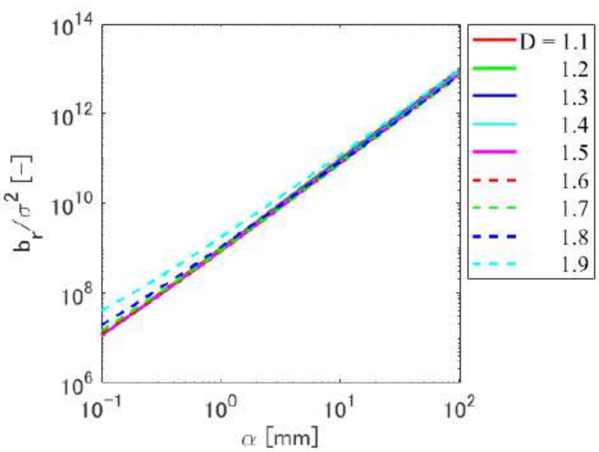
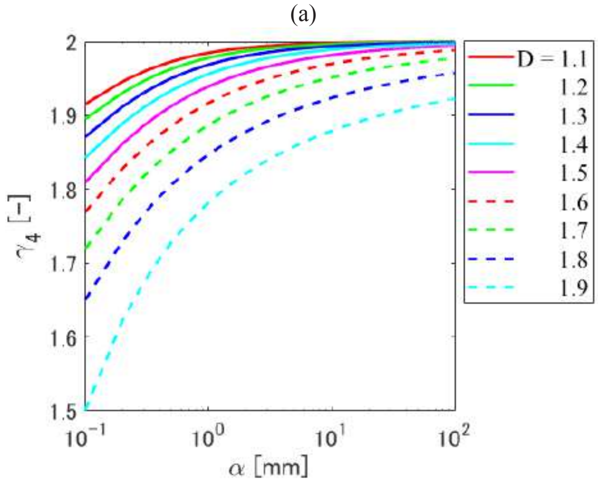


(b)

Fig 3.(a) Logarithmic plot of A_c as a function of α in the entire range of D, and (b) its local slope γ_3.



(a)



(b)

Fig 4. (a) Logarithmic plot of b_r/σ^2 as a function of α in the entire range of D, and (b) its local slope γ_4.

5 Conclusion

We reported the statistics of the derivatives of the intensity and phase in fractal speckles. To obtain these statistical properties, we derived the two key parameters in the joint density function of speckle intensity, phase and their derivatives with respect to x and y. The two parameters were given by the integrals of the intensity distributions of the scattering spot obeying a negative power-law function. We also derived the correlation area in fractal speckles, which is related to the two parameters in the statistics of the derivatives of speckle intensity and phase. To represent the realistic model of the intensity distribution of the scattering spot for producing fractal speckles, the Fisher-Burford and Gaussian approximations were applied to the negative power function, in which the former is to adjust the profile of negative power function around the origin and the latter is to limit the extent of the intensity distribution of the scattering spot. The results showed that the two parameters and the correlation area vary monotonically as the extent of the scattering spot increases, depending on the power exponents of intensity distributions of scattering spots unlike ordinary speckles. In conclusion, the results in the present study would play an important role in some problems related to fractal speckles, such as the properties of local maxima and the level crossing problems of intensity, the statistical properties of the geometrical ray directions, the density of vortex and so on.

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