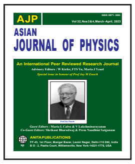

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The eikonal equation via adaptive control theory

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Dedicated to Professor Jay M Enoch.

The eikonal equation can be considered as the fundamental equation in geometric optics. It is demonstrated here that the equation can be derived as a control theoretic problem using the method of dynamic programming.

Keywords: Geometric optics, Eikonal equation, Adaptive Control, Dynamic programming, Fermat's principle.

1 Introduction

The fundamental equation of geometric optics is the eikonal equation. In the geometric optics approximation, the propagation of electromagnetic waves (in the optical range) can be represented as the transfer of wave energy which are described by geometric relations. The eikonal equation describes the constant phase surfaces of a wave and the propagation of light energy along a certain direction and leads to the concept of a ray of light (see Ref [1]; appendix A). There is an elegant statement that "*classical mechanics corresponds to geometric optics limit of wave motion*" in which light rays orthogonal to wavefronts correspond to particle trajectories orthogonal to surfaces of constant phase [2]. There are various ways to derive the eikonal equation (e.g., using Fermat's principle of least time, or using Euler-Lagrange equations) and here we use the techniques of adaptive control theory to obtain the equation.

Consider a deterministic sequential physical process, that is if f be an arbitrary quantity having an initial value f_0 at time t_0, f_1 at time t_1, f_2 at time $t_2, \ldots$, f_n at time t_n with $t_0 < t_1 < t_2 \ldots < t_n$, then it constitutes a multi-stage linear sequential process, and we also note that f_r is a function of f_{r-1} only. Then, we can reduce the process to one dealing with subprocesses and this functional $f_r = f_r\,(r-1)$ can be used to solve complicated physical problems. We need to define the functional f_r from which the process (a state variable) can be characterized uniquely. Optimizing this functional is equivalent to optimization of the entire process. For this optimization, we use the technique of dynamic programming and the principle of optimality introduced by Richard Bellman [3]. This principle states that an optimal policy has the property that whatever the initial state and initial decision are the remaining decisions must constitute an optimal policy with regard to the state resulting from the initial decision. In this definition, the decision is nothing but the selection of a single value of all possible values of f_0 of the arbitrary function. When applying this, we require that the system be closed, that is be in equilibrium. The method is a flexible tool for optimizing the behavior of the linear system. Dynamic programming can be applied to any arbitrary time varying system in order

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to predict the final state of the associated sequential process. The reader is referred to many excellent texts on dynamic programming e.g., [3-5]. This technique has been applied over the decades to a wide range of problems- from economics, signal and image processing to computational biology and genomics in addition to physics. In the field of optical sciences, Robert Kalaba was the first to apply the principle of optimality to derive the eikonal equation [Ref 6; see also Ref 7]. This analysis was applied to the case of an optical waveguide [8] and further developed to case of arbitrary inhomogeneous periodically pooled waveguides [9]. Brandsttter extended this to the case of inhomogenous media [10]. Other applications in optics are summarized in the chapter by Calvo, Perez-Rios and Lakshminarayanan [11]. In this paper, we derive the eikonal cast in the language of control theory.

2 The eikonal equation by a control theoretical method

Consider a control system

$$\dot{X} f(x, u)\ x \in \mathbb{R}^n,\ u \in U \subset \mathbb{R}^k$$

The task is to minimize a feedback control $u(x, t)$ $0 \le t \le t_1$ that minimizes the integral of the form

$$J[u(o)] = \int_t^{t_1} r\big(x(\tau), (u)\big)\, d\tau + g\big(x(T)\big)$$

for each $t \in [0, t]$ such that

$$X(t_{(1)}) = P$$

where P is a function in $\mathbb{R}^n$.

Let $V(x, t)$ be a cost function

$$V(x, t) = \min_{u(\tau) < \tau \le t_1} J[u(\cdot)]$$

From the method of dynamic programming, $V(t)$, solves the so-called Hamilton-Jacobi Bellman Equation

$$\frac{\partial J(x, t)}{\partial t} + \min_{v \in U} \left\{ \frac{\partial v(x, t)}{\partial x} \cdot f(x, v) + \gamma(x, v) \right\} = 0$$

with boundary condition

$$[X(t_1), t_1] = g\big(x(t_1)\big)$$

The minimizing function $u(x, t)$ is an optimal control and given by

$$u(x, t) = \operatorname{Arg} \min_{v \in U} \left\{ \frac{\partial V(x, t)}{\partial x} \cdot f(x, v) + \gamma(x, v) \right\}$$

The corresponding optimal trajectory $X(t)$ can be obtained by integrating the ordinary differential equation

$$\dot{X} = f\big(x, u(x, t)\big),\ x(t_1) = P$$

With this background formulation let us now derive the eikonal equation.

Consider the case in $\mathbb{R}^2$ and use (x, z) as coordinates. Although the physical phenomena described by the eikonal is not a control system, we can change it into a control problem and apply the dynamic programming technique given above. Let us change t into s (the arc length) and we can control the direction of velocity.

The control system now is

$$X' = \cos u$$

$$Z' = \sin u$$

where u is a control variable and $'$ represents differentiation with respect to arc length s. With these modifications the problem becomes one of minimizing

$$\int_a^b n\left(x(\tau), z(\tau)\right)\delta\tau + 0$$

with $c = 1$, such that the constant of integration is

$$\left(X(b), z(b)\right) = (x_0, y_0)$$

where b is a free parameter since total arc length is not prescribed.

Let $T(x, z, s)$ be the time to go (i.e., cost) function instead of, $V(x, z, s)$. Since b is free, the corresponding transversality condition is $T(x, z, s)$, which is independent of s. The Hamilton-Jacobi-Bellman equation now becomes for the time-to-go function $T(x, z)$.

$$0 + \min_{v \in \mathbb{R}} \left\{\frac{\partial T(x, z, s)}{\partial x}\cos v + \frac{\partial T(x, z, s)}{\partial x}\sin v + n(x, z)\right\} = 0$$

with $T\left(x(b), y(b)\right) = 0$

From the above equation the optimal control should be such that

$$\cos u(x, z) = \frac{-T_x}{\sqrt{(T_x{}^2 + T_z{}^2)}} + n(x, z) = 0$$

$$T_x{}^2 + T_z{}^2 = n^2(x, z)$$

The above equation is recognizable as the fundamental equation of geometric optics, the Eikonal.

From the solution of the eikonal, we can solve for b and arrive at

$$T\left(x(b), z(b)\right) = n^2\,(x_0, z_0)$$

And the optimal trajectory is

$$X' = \frac{-T_x}{\sqrt{(T_x{}^2 + T_z{}^2)}}$$

$$Z' = \frac{-T_z}{\sqrt{(T_x{}^2 + T_z{}^2)}}$$

For $0 \le s \le b$, we arrive at

$$\left(x(b), z(b)\right) = (x_0, z_0)$$

3 Discussion

The derivation of the eikonal equation can be extended. Consider a function $\mathbb{R}^3$, in this case the functions control system is $u_1, u_2, u_3 \in \mathbb{R}$ satisfies

$$u_1{}^2 + u_2{}^2 + u_3{}^2 = 1$$

and the trajectory can be parametrized by the arc length s.. Alternatively, we can use

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \cos u_1 \, \cos u_2 \\ \sin u_1 \, \sin u_2 \\ \sin u_2 \end{pmatrix}$$

where there is no constraint on the two controls u_1 and u_2.

Another variant is instead of giving a final point P, a final set $(x(T), z(T)) \in s$, where s is arc length in $\mathbb{R}^2$ can also be considered. In this case there will be a corresponding transversality condition. Also, the total arc length b can be prescribed instead of it being a free variable.

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