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Bell state discrimination and error correction

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An ancilla-based single particle measurement scheme is illustrated for Bell state discrimination which does not lead to the collapse of the wave function. It enables the extraction of partial information, e.g., parity and phase independently, that generalizes to multi-particle entangled states of qubits, which also applied to qudits. It can be used for error correction in a quantum circuit architecture involving basic gates like Hadamard and Controlled-NOT. It helps unravel the syndrome operators and, what is more, in the multi-partite state that the individual constituents lack.

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1 Introduction

Entangled states play a key role in the transmission and processing of quantum information [1,2]. Using an entangled channel, an unknown state can be teleported [3] with local unitary operations, appropriate measurement and classical communication; one can achieve entanglements wapping through joint measurement on two entangled pairs [4]. Entanglement leads to an increase in the capacity of the quantum information channel, known as quantum dense coding [5]. Maximally entangled bipartite Bell states offer the clearest illustration of these concepts, while multipartite entangled states such as the GHZ and W states are increasingly utilized in diverse quantum information processing tasks [6,7].

Bell states are the simplest examples of maximally entangled states that have been familiar from atomic physics and spectroscopy in the form of singlet and triplet states. They have formed the corner stone in the optical investigations of the quantum nonlocality, leading to the Nobel prize for Aspect, Zellinger and Clauser in 2022 [8-12].

The nonlocality of these states follows from the single particle measurement,which leads to the collapse of the wave function. Following the vocabulary of quantumin formation, the singlet Bell state, shared between Alice (A) and Bob (B). $|\phi^-\rangle = \frac{1}{\sqrt{2}}\left[|0_A1_B\rangle - |1_A0_B\rangle\right]$ is intrinsically nonlocal as Alice and Bob can be spatially separated to any extent. Here, $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix} = |\uparrow\rangle$, $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix} = |\downarrow\rangle$, are the conventional igenstate of the Pauli matrix $\sigma_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ with eigenvalues +1 and −1, respectively. The state $|\phi^-\rangle$ is non separable into product states of the individual constitutes and in fact it is maximally entangled state [13]. Single particle measurement at Alice's end, for example, will lead to the collapse of the state to $|0_A1_B\rangle$ or $|1_A0_B\rangle$, depending on the measurement outcome of Alice's state being $|0\rangle_A$ or $|1\rangle_A$, respectively.This is independent of the

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spatial location of Bob and is at the heart of the EPR paradox [14] involving quantum nonlocality and hidden variable theories [15]. The entangled states cannot only collapse under single particle measurement but can also collapse under the decohering infuence of the environment. Entangled states are integral components of the quantum computation platform and communication protocols.

Making use of single-qubit operations and the Controlled-NOT gates, one can produce various entangled states in a quantum net work [1]. It may be of interest to know the type of entangled state that is present in aquantum network, at various stages of quantum computation and cryptographic operations, without disturbing these states. Nonorthogonal states are impossible to discriminate with certainty [16]. A number of results have recently been established regarding distinguishing various orthogonal Bell states [17-20]. It is counter intuitive to know that multipartite orthogonal states may not be discriminated with only local operations and classical communications (LOCC) [17]. However, any two multipartite orthogonal states can be unequivocally distinguished through LOCC [19]. If two copies belonging to the four orthogonal Bell states are provided, LOCC can be used to distinguish them with certainty. It is not possible to discriminate, either deterministically or probabilistically, the four Bell states if only a single copy is provided. It is also known that any three Bell states can not be discriminated deterministically if only LOCC is allowed.

It has been proven that a perfect Bell measurement can not be achieved using only linear elements [21]. A number of theoretical and experimental results already exist in this area of unambiguous state discrimination [22-24]. Appropriate unitary transformations and measurements, which map the Bell states onto disentangled basis states, can unambiguously distinguish all four Bell states [23-25]. However, in the process of measurement the entangled state is altered. This is not an issue if the Bell state is not required further in the quantum net work. We present in this article a scheme which discriminates all the four Bell states deterministically and is able to preserve these states for further use.

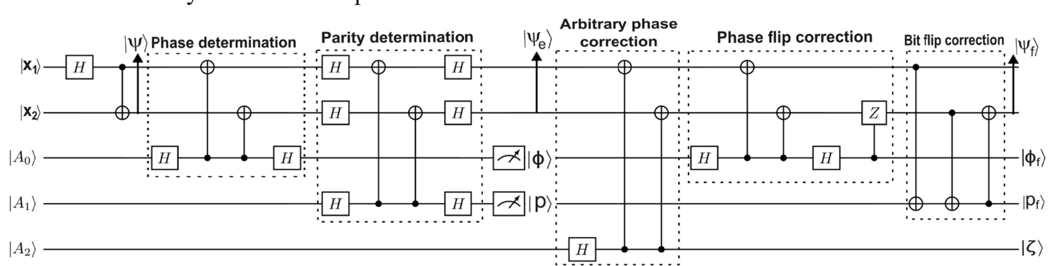

Fig 1. Schematic diagram of quantum circuit for Bell state discrimination and error correction.

Signifcant research is being carried on to achieve practical quantum computing, particularly with a focus on addressing computational errors caused by noise, decoherence in quantum processes.The tendency of maximally entangled states to degrade their entanglement when transmitted through quantum channel in quantum circuit gives birth the idea of the development of methods to address arbitrary phase change, phase flip and also bit flip error, all of which need to be corrected for error free accurate outcome. Encoding methods of error-correcting codes at an optimized level have been proposed [26,27] to boost the quantum process or in noisy channels. Even single bit-flip or phase-flip error or their combination have been detected experimentally in trapped ion qubits [28]. An encoding scheme of quantum polar codes on one logical qubit is proposed to devise a factory-based fault-tolerant quantum computer [29,30]. Nondestructive discrimination of generalized maximally entangled states [31,32] and their experimental realization for higher qubits and qudits also [33,34] has been explored. Recently, it is reported that non destructive discrimination can be done with distant parties where ancillas are taken as entangled states, overlooking the need for local operation [35]. Among the various error correction strategies in quantum net work protocols, here automated error correction is our focus where the information of the outcome states encoded on the ancillas are used. Methods of this kind of error correction for bit-flip or phase-flip errors have been shown [36,37]. We also demonstrate here a method of automated error correction if there any error present in the discriminated Bell states while it is continuing through quantum channel.

2 Results and Discussions

As LOCC alone is insufficient for discriminating all the four Bell states deterministically, we will make use of two ancilla bits, along with the entangled channels.Throughout the protocol, we will only employ local unitary operations. At the end, measurements are carried out on the two ancilla bits and the Bell states for further operations.

It should be noted that the Bell states:

$$|\psi_1\rangle = \frac{1}{\surd 2}\left[|0_A 0_B\rangle + |1_A 1_B\rangle\right]$$

$$|\psi_2\rangle = \frac{1}{\surd 2}\left[|0_A 0_B\rangle - |1_A 1_B\rangle\right]$$

$$|\psi_3\rangle = \frac{1}{\surd 2}\left[|0_A 1_B\rangle + |1_A 0_B\rangle\right]$$

$$|\psi_4\rangle = \frac{1}{\surd 2}\left[|0_A 1_B\rangle - |1_A 0_B\rangle\right], \tag{1}$$

when operated on by single qubit operators such as Hadamard and Pauli matrices, separately or in combination, get transformed into each other.This is an interesting property which can easily transform one Bell state to another on demand. This property of Bell states proves very handy in distinguishing them. Exploiting the above nature of the Bell states, we have designed a circuit for the Bell state discrimination, as shown in (Fig 1). It consists of two quantum channels, depicted as $|x_1\rangle$ and $|x_2\rangle$, carrying the entangled state $|\Psi\rangle$, which have to be discriminated; two ancilla qubits A_0 and A_1 are used for carrying out local operations.In the end, measurement is taken on these ancilla bits to know with certainty, the type of Bell state that exists in the channel. Measurement on the first ancilla will differentiate the four Bell states into two pairs i.e., either $|\psi_1\rangle/|\psi_3\rangle$ or $|\psi_2\rangle/|\psi_4\rangle$ as given in Eq (2). While the measurement on the second ancilla differentiates the Bell states within these two groups as stated in Eq (3).The remarkable property of this circuit is that, the Bell states in first two quantum channels retain their initial states, even after being discriminated. Here, we have used the Hadamard operation on the entangled channel while differentiating the Bell states in Eqs (2) and (3), though one can also use other suitable single-qubit operations. In Table 1, we have shown the results of the measurements on both the ancillas when different Bell states are present in the given circuit (Fig 1). Before the measurement, the states can be explicitly written as,

$$|RA_0\rangle = \left[I_2 \otimes I_2 \otimes H\right] * \left[(x_1 \oplus A_0) \otimes (x_2 \oplus A_0) \otimes I_2\right] * \left[I_2 \otimes I_2 \otimes H\right] * \left[|\Psi\rangle \otimes |A_0\rangle\right] \tag{2}$$

$$|RA_1\rangle = \left[\mathrm{H}^{\otimes 3}\right] * \left[(x_1 + A_1) \otimes (x_1 + A_1) \otimes I_2\right] * \left[\mathrm{H}^{\otimes 3}\right] * \left[|\Psi\rangle \otimes |A_1\rangle\right]. \tag{3}$$

Table 1. States and corresponding measurement results

Bell State	Measurement A_0	Measurement A_1
$\lvert\psi_1\rangle$	0	0
$\lvert\psi_2\rangle$	1	0
$\lvert\psi_3\rangle$	0	1
$\lvert\psi_4\rangle$	1	1

Discriminated Bell states also have been subjected to the error correction procedure to confrm their error-free nature. Regenerating phase measuring ancilla $|\phi\rangle$ and parity measuring ancilla $|p\rangle$, next procedure for error correction is demonstrated. Here, three steps of error correction have been followed i.e., arbitrary phase correction, phase-flip correction and bit- flip correction. Here, it is noteworthy to mention that arbitrary phase change can transform the non-maximally entangled state to the maximally entangled state, but phase-flip orbit-flip keep unchange the nature of the state i.e., maximally entanglement, although changes into different state.

2.1 Arbitrary phase change correction

For arbitrary phase correction, we have taken an extra ancilla $|A_2\rangle = |0\rangle$. After applying an arbitrary phase correction to the erroneous state $|\psi_e\rangle$, one cannot guarantee the complete elimination of phase-flip errors, as a nonzero probability of residual errors may still remain. The phase error of the maximally entangled state is transferred to the ancillary qubit through the application of a CNOT gate.The process can be written as,

$$|\psi_e\rangle\,|\zeta\rangle = [I_2^{\otimes 2} \otimes H_{A_2}]*[(x_1 \oplus A_2) \otimes (x_2 \oplus A_2) \otimes I_2]*[|\psi_e\rangle\,|A_2\rangle]. \quad (4)$$

As an example, for the Bell state with arbitrary phase error $|\psi_1\rangle = \frac{1}{\sqrt{2}}[|0\,0\rangle + e^{in}\,|11\rangle]$, where $e^{i\eta}$ is the arbitrary phase. This procedure can be explained as follows

$$|\psi_1\rangle\,|A_2\rangle = \frac{1}{\sqrt{2}}[|0\,0\rangle + e^{in}\,|11\rangle]\,|0\rangle$$

$$= \frac{1}{\sqrt{2}}[|0\,0\rangle + e^{in}\,|11\rangle]\,(|0\rangle + |1\rangle)$$

$$= \frac{1}{\sqrt{2}}[|0\,1\rangle + |10\rangle]\,(|1\rangle + e^{in}\,|0\rangle)$$

Controlled Pauli-X operation helps to shift arbitrary phase error into the ancilla state, leaving the Bell state free from arbitrary phase error.

2.2 Phase-flip correction

In this segment of the circuit, phase-flip error of the state $|\psi_{e1}\rangle$ is checked, if there any error exist then at the end controlled unitary Pauli-Z gate recover it. In the phase-flip correction circuit, if the initial phase qubit and final phase qubit of the state are diferent, which conflrms the occurrence of the phase-flip error, the phase measurement ancilla would be $|1\rangle$. On the contrary, if no bit-flip error occurs, the phase measuring ancilla or phase qubit will end up in the state $|0\rangle$, confrming that the state is phase-flip error-free. The procedure can be written as

$$|\psi_{e1}\rangle\,|\phi_f\rangle = [(x_1 \oplus A_0) \otimes [(x_2 \oplus A_0) * (x_2{}^Z \oplus A_0\,] \otimes I_2] \otimes [|\psi_{e1}\rangle\,|\phi\rangle], \quad (5)$$

where $x_2{}^Z$ is the Controlled Pauli-Z gate operated on $|x_2\rangle$ and $|\psi_{e1}\rangle$ is corrected state if there exist any phase-flip error.

2.3 Bit-flip correction

When an erroneous state $|\psi_{e2}\rangle$ passing throughBit-flip correction portion of the circuit where bit-flip errors are detected, then comparing the initial and fnal parity qubits (i.e., parity measuring ancilla), bit-flip correction can be solved. The process can be represented as,

$$|\psi_{e2}\rangle\,|pf\,\rangle = |I_2\rangle \otimes (x_2 \otimes A_1) \otimes [(A_1 \oplus x_1) * (A_1 \oplus x_2)] *[|\psi_{e2}\rangle|p\rangle]. \quad (6)$$

If a bit-flip error occurs, the fnal parity-measuring ancilla will be $|1\rangle$; otherwise, it will be $|0\rangle$, bearing the confrmation about the error-free state.

There are far well-known syndrome operators which stabilize codes for a lucid illustration of the operation for error correction on the quantum polar codes; the interested readers are referred to the references [29,30]. The syndrome operators $X = \sigma_x \otimes \sigma_x$, $Y = \sigma_y \otimes \sigma_y$ and $\Sigma = \sigma_z \otimes \sigma_z$, where σ_x, σ_y and σ_z are the Pauli matrices, satisfy the commutation relations $[X,Y] = 0$, $[X, \Sigma] = 0$ and $[Y, \Sigma] = 0$. It is well-known that the Bell states are eigen functions of these operators.They have an eigen value of +1 for the following states: $X|\psi^+\rangle$, $X|\phi^+\rangle$, $Y|\psi^-\rangle$, $Y|\phi^+\rangle$, $\Sigma|\psi^+\rangle$ and $\Sigma|\psi^-\rangle$. Conversely, they have an eigen value of –1 for the states: $X|\psi^-\rangle$, $X|\phi^-\rangle$, $Y|\psi^+\rangle$, $Y|\phi^-\rangle$, $\Sigma|\phi^+\rangle$ and $\Sigma|\phi^-\rangle$. It is worth mentioning that $|\phi^-\rangle$ is the only eigen state of all the above operators, i.e.,X, Y and Σ, with same eigen value −1.

3 Conclusion

We have illustrated the discrimination of Bell states and presented a method to address potential errors in an automated manner that may happen during their circulation through a quantum channel.These errors arise due to noise in the channel and the effects of system decoherence. Basically, two ancillas are used to discriminate the Bell states deterministically, along with the measurement of phase-fiperror and bit-fiperror. An extra ancilla is utilized to ensure arbitrary phase error correction.The key advantage of the ancilla-based approach is its ability to perform error correction with minimal disruption to the quantum circuit, as entangled states maintain their identity.This approach keeps the quantum system reliable, protects the integrity of the quantum information it holds and makes it capable of flxing errors easily. Reducing these errors, helps robust quantum operation which is essential for the development of quantum technologies.This method extends to qudits and to distributed quantum networks.

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