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Vertex algebras and Hodge structures (Beilinson-Bernstein correspondence)

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Dedicated to Prof Daniel Malacara-Hernández

This article surveys the connections between Hodge structures and vertex algebras in conformal field theory and is expository text. These structures appear in various contexts of theoretical and mathematical physics. Hodge structure is a generalized complex structure which appears on solution space of integrable systems. Vertex algebras, as highest weight representations of infinite-dimensional Lie algebras, provide a frame work that intersects with both mathematical and physical theories. We explain the connection between Hodge structures and theory of highest weight modules over affine algebras by a generalization of the Beilinson-Bernstein correspondence. The Beilinson-Bernstein localization draws parallels between the variation of Hodge structures and highest weight modules over flag manifolds of semisimple Lie groups. This framework has profound implications for quantum field theory, where the structure of vertex algebras influences the understanding of symmetries and interactions in conformal field theories. We also consider a broader version of the Bernstein correspondence within the context of the geometric Langlands correspondence over local manifolds. Vertex algebras and representations of a fine algebras provide a powerful frame work for describing symmetry and state evolution in physics, optics, and information technology. They under-pin conformal field theories in quantum physics, model photon correlations, entanglement in quantum optics, and enhance quantum information processing. These structures offer a unified approach to analyzing complex systems across these fields. doi.10.54955.AJP.33.12.2024.843-864

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1 Introduction

We study the relationship between Hodge theory as a structure for integrable systems over complex numbers and some aspects of theoretical quantum sciences. Hodge structure is a generalized complex structure on the solution space of integrable systems over complex numbers. In this way, it appears as an additional structure in solving different differential systems. We also study vertex algebras, which are realized as highest weight representations of infinite-dimensional Lie algebras. These algebras play a crucial role in understanding the symmetries and interactions governing multicomponent particle systems. We discuss several correspondences connecting classical Hodge theory to the theory of representations of affine algebras. The correspondences relates the two theories and their symmetries.

Our analysis goes through the Beilinson-Bernstein localization, that links the variation of Hodge structures with highest weight modules over flag manifolds of semi simple Lie groups. This connection has implications with the underlying symmetries in quantum theories, where group theoretical methods

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are describing the behavior of multi-component systems. We also introduce the geometric Langlands correspondence, which provides a more generalized framework for understanding this connection. Our analysis enables a deeper understanding of quantum systems in theoretical physics.

Vertex algebras provide a powerful tool to describe integrable systems in physics, including optics, or any theory with a quantum description for a many particle system. This method is universal, and applies to theories like particle physics, optics, and computer sciences. Hodge structure is a generalized complex structure which can appear on the solution space of an integrable system over complex numbers. We deal with correspondences that connect the classical Hodge theory with the theory of representations of infinite dimensional Lie algebras. Vertex algebras and vertex operators provide a powerful tool in describing these theories on the representation side.

Vertex algebras have significant applications in quantum physics and optical sciences, for instance in the description of photon correlations. The mathematical structure can be instrumental in describing optical processes, such as wave propagation in structured media. The conformal symmetry inherent in vertex algebras can model wave propagation in photonic crystals or metamaterials with periodic or quasi-periodic structures. In this context vertex algebras are fundamental to describe certain quantum states in interacting many particle systems.

We briefly introduce the context below. Later in the text we make it more specific.

1 *Vertex Operator Algebras*

Vertex algebras naturally emerge from the highest weight representations of affine or Virasoro algebras. To establish a basic frame work, let us consider

$$D = t\,\frac{d}{dt}$$

as a differential operator acting on the ring $R = \mathbb{C}[t, t^{-1}]$. The Lie algebra $\mathfrak{g}$ of derivations of R is generated by $t^m D$, $m \in \mathrm{Z}$. This algebra is graded, where $t^m D$ is assigned weight m. We are particularly interested in central extensions

$$0 \to \mathbb{C}.c \to V \to \mathfrak{g} \to 0,$$

which define the Virasoro algebra. This is also a graded Lie algebra. Specific highest weight representations of V play a role in conformal field theory. These representations are such that, for an appropriate choice of $\widehat{D}$ extending D within the central extension, the character

$$\mathrm{Trace}\left(q^{\widehat{D}}\right)$$

is well-defined and corresponds to the q-expansion of a modular form, [1]. The concept of the Virasoro algebra can be generalized to that of a vertex algebra. We can view the Virasoro algebra V as a vector space acted upon by commuting operators v_n, where V is generated by $1 \in V$ and the action of these operators. Consequently, V becomes a commutative ring where 1 serves as the identity, and the actions of all operators correspond to multiplication within V. We define an operator

$$\phi(t) : v \to \sum_i D^i v \cdot t^i / i!, \quad V \to \mathrm{End}(V)\,[t, t^{-1}]$$

called the vertex operator. The maps

$$\mathrm{Trace}\,(\phi(x)\,\phi(y)\ldots)$$

are referred to as correlation functions, drawing their name from their analogs in quantum field theory. The structure of a vertex operator algebra is intended to explain a conformal infinitesimal deformation of V.

A vertex operator algebra is defined by a 4-tuple $(V, Y, 1, \omega)$, where V is a $\mathbb{Z}$-graded vector space equipped with a linear map

$$Y(.,z): V \to \mathrm{End}(V)[[z, z^{-1}]], \qquad Y(v, z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1},$$

where $1 \in V_0$ is the vacuum vector satisfying $Y(1, z) = id_V$ and $v_{-1}1 = v$. The vector $\omega \in V_2$ is known as the conformal or Virasoro element, such that

$$Y(\omega, z) = \sum_n L(n) z^{-n-1}$$

provides a collection of Virasoro generators $L(n)$ where $L(0)|_{V_n} = n \cdot id_{V_n}$. The operator L(−1) satisfies

$$[L(-1), Y(v, z) = \frac{d}{dz} Y(v, z),$$

and the Jacobi identity for the vertex operator Y is assumed, cf. [2]. We focus on vertex algebras of CFT-type, meaning $V_n = 0$ for $n < 0$ and $V_0 = \mathbb{C}.1$. If V is generated by a subset $S \subset V$, then

$$V = \mathrm{span} \qquad \{ v^1_{n_1} \dots v^k_{n_k}.\mathbf{1} | \} v_i \in S\}.$$

A unitary vertex operator algebra is one equipped with a positive definite Hermitian form. This concept extends to modules over these algebras via an anti-involution. An anti-linear automorphism is a map defined as

$$\phi : V \to V, \qquad \phi(1) = 1,\ \phi(\omega) = \omega,\ \phi(u_n v) = \phi(u)_n \phi(v) \qquad \forall u, v \in V.$$

Unlike Lie algebras, defining an invariant bilinear form on a vertex algebra is notably more complex. For instance, the contragredient (V', Y') of a vertex algebra module (V, Y) is defined using the form

$$\langle Y'(v, x) w', w\rangle = \langle w', Y(e^{xL(1)}(-x^2)^{L(0)} v, x^{-1}) w\rangle, \qquad v, w \in V,\ w' \in V'.$$

The form must also account for the invariant bilinear form on the vertex algebra. A unitary vertex algebra includes a positive definite Hermitian form. For such an algebra, the positive definite Hermitian form

$$(.,.)_{\mathrm{unitary}} : V \times V \to \mathbb{C},\ \exists\, \lambda \in \mathbb{C};\ (u, v) = \lambda(1,1)$$

is uniquely specified by its value at $(1,1)$, a fact easily shown using the axioms of a vertex algebra and the invariance property.

Vertex algebras and their homomorphisms form a tensor category. This means we can tensor finitely many vertex algebras

$$\bigotimes_{i=1}^{p} (V_i, Y_V, I_i, \omega_i),\ \omega = \omega_1 \otimes 1 \otimes \dots \otimes 1 + \dots + 1 \otimes 1 \otimes \omega_p$$

The structure of a vertex algebra must adhere to a locality axiom. Although, its detailed description is outside the scope of this note, we refer readers to [3] for a complete discussion.. Briefly, this means that for any $A, B \in V$, the two formal power series in two variables obtained by composing $Y(A, z)$ and $Y(B, w)$ in both possible orders are equal, possibly after multiplying by a sufficiently large power of $(z - w)$. Formally,

$$(z-w)^N [Y(A, z), Y(B, w)] = 0, \qquad \text{for some } N \in \mathbb{Z}_+.$$

In this context, we define a normally ordered product as :

$$A(z)\, B(\omega) := -\sum_m \left\{ \sum_{m<0} A_m B_n z^{-m-1} + \sum_{m\geq 0} A_m B_n z^{-m-1} \right\} \omega^{-n-1}$$

for vertex operators. This product can be inductively extended to more than two factors.

Remark 1.1 [4]. *In the case of holomorphic vertex algebras, where the operators $V(a,x)$ are holomorphic, they correspond to commutative rings with derivations. The notation $V(u, z)\, v$ can be interpreted as a deformation of $u^z . v$. If we begin with a commutative algebra equipped with a derivation D, we can define the vertex operator as:*

$$V(a, x)\, b = \sum_{i\geq 0} \frac{(D^i a) b x^i}{i!}. \tag{1}$$

Conversely, if V is a vertex algebra, we can recover the algebra and derivation structure via the definitions:

$$ab = V(a, 0)\, b,$$

and

$$Da = \text{coefficient of } x^1 \text{ in } V(a, x)\, b.$$

In this framework, we can introduce a new notation:

$$a^x = \sum_{i \geq 0} \frac{x^i D^i a}{i!}$$

Using this, the vertex operation becomes:

$$a^x b = \sum_{i \geq 0} \frac{x^i D^i (ab)}{i!} \tag{2}$$

Here, x is treated as an element of the formal group $\widehat{G}_a$. The formal group $\widehat{G}_a$ has a formal group ring $H = \mathbb{C}[D]$, and its coordinate ring is the ring of formal power series $\mathbb{C}[[x]]$. The tensor category of modules with derivations corresponds to the category of modules over the formal group ring H. Thus, holomorphic vertex algebras can be viewed as commutative ring objects in this category.

In the non-holomorphic case, however, the expressions $a^x b^y$ (for $a, b \in V$) are no longer holomorphic and may exhibit singularities. Nevertheless, the identities of vertex algebra theory become easier to interpret using the new notation. For example:

$$V(a, x)\, b = e^{xL_{-1}}\big(V(b, -x)\, a\big) \qquad \Leftrightarrow \qquad a^x b = (b^{x^{-1}} a)^x. \tag{3}$$

An intertwining operator between three modules $(W_1, Y_1), (W_2, Y_2)$, and (W_3, Y_3) is a linear map:

$$I(.,z) : W_2 \to \mathrm{Hom}(W_3, W_1)\{z\}, \qquad u \to I(u, z) = \sum_{n \in \mathbb{Q}} u_n z^{-n-1}, \tag{4}$$

which satisfies certain compatibility conditions. The vertex operator $Y_M(.,z)$ can be viewed as a special case of an intertwining operator, where $W_3 = W_1 = M$. One definition of these operators is:

$$I(w, z)v = e^{zL(-1)} Y_M(v, -z)w. \tag{5}$$

Intertwining operators play a crucial role in defining a product structure in the category of vertex algebras. The standard tensor product of two Lie algebra modules is generally not a module, complicating the construction of an associative product structure in the tensor category of vertex operator algebras (VOAs). Intertwining operators also provide the foundation for the theory of conformal blocks. Using the $L(-1)$-property and intertwining operators, one can derive a system of differential equations whose solutions correspond to these modules. These are D-modules, acted upon by certain differential operators [5].

2 *Variation of Hodge structure:*

A polarized Hodge structure on a $\mathbb{Q}$-vector space V is defined by a representation

$$\phi : \mathbb{U}(\mathbb{R}) \to \mathrm{Aut}\,(V_{\mathbb{R}}, Q), \qquad \mathbb{U}(\mathbb{R}) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \qquad a^2 + b^2 = 1.$$

The group $G_{\mathbb{R}} = \mathrm{Aut}(V_{\mathbb{R}}, Q)$ is a real simple Lie group. The period domain D associated with the Hodge structure ϕ is the moduli space of polarized Hodge structures on the vector space V with fixed Hodge numbers. The group $G_{\mathbb{R}}$ acts transitively on D by conjugation:

$$D = \{\phi : S^1 \to G_{\mathbb{R}} \mid \phi = g^{-1} \phi_0\, g\}.$$

The isotropy group H of a reference polarized Hodge structure (V, Q, ϕ) is a compact subgroup of $G_{\mathbb{R}}$ and contains a compact maximal torus T. The Lie algebra $\mathfrak{g}$ of the complexified simple Lie group $G_{\mathbb{C}}$ is a $\mathbb{Q}$ -linear subspace of End (V). The bilinear form Q induces a non-degenerate symmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g}$

$\to \mathbb{C}$, which, up to scale, is the Cartan-Killing form tr(ad(x) ad(y)). For every point $\phi \in D$, the representation

$$\mathrm{Ad}(\phi) : \mathbb{U}(\mathbb{R}) \to \mathrm{Aut}(\mathfrak{g}_{\mathbb{R}}, B)$$

defines a Hodge structure of weight 0 on $\mathfrak{g}$, polarized by B.

For each nilpotent transformation $N \in \mathfrak{g}$, one can define a limit mixed Hodge structure. The local system $\mathfrak{g} \to \Delta^*$ is equipped with monodromy $T = e^{\mathrm{ad}(N)}$ and a Hodge filtration defined with respect to a multivalued basis of $\mathfrak{g}$ by

$$e^{\log(t)} \frac{N}{2\pi i} F^{\bullet}$$

where $F^{\bullet}$ is the natural Hodge filtration on $\mathfrak{g}$. This yields a limit mixed Hodge structure $(\mathfrak{g}, F^{\bullet}, W(N).)$. The polarizing form B defines perfect pairings:

$$B_k : Gr_k^{W(N)} \mathfrak{g} \times Gr_{-k}^{W(N)} \mathfrak{g} \to \mathbb{Q},\ B_k(u, v) = B(u, N^k v),$$

via the hard Lefschetz isomorphism $N^k : Gr_{-k}^{W(N)} \mathfrak{g} \cong Gr_k^{W(N)} \mathfrak{g}$.

Now consider a family of projective manifolds defined by a proper smooth map:

$$f : X \to S, \qquad X_s = f^{-1}(s),$$

where S is a quasi-projective variety. This induces a polarized variation of Hodge structures (VHS):

$$(V = R^k f_* \mathbb{C}, F^{\bullet}).$$

If dim S = 1, this variation corresponds to a topological deformation of the Hodge structure $V = H^k(X_s, \mathbb{C})$ over a punctured disc. Understanding the asymptotic behavior of VHS is central in Hodge theory. Assume that V is equipped with a limit Hodge filtration. By the Riemann-Hilbert correspondence, a local system of Hodge structures defines a D-module with a flat connection on S. This leads to a decreasing filtration $F^{\bullet} = (F^i)$ on the vector bundle $\mathrm{V} \otimes_{\mathbb{Q}} \mathrm{O}_S$ by holomorphic sub-bundles, along with a flat connection:

$$\nabla : \mathrm{V} \otimes_{\mathrm{Q}} \mathrm{O}_S \to \mathrm{V} \otimes_{\mathrm{Q}} \Omega^1{}_S,$$

satisfying Griffiths transversality:

$$\nabla(F^i\, \mathrm{V}) \subset F^{i-1} V \otimes \Omega^1_S.$$

Additionally, the data include a flat bilinear pairing:

$$P : V \times V \to \mathbb{Q}.$$

In the context of conformal field theory, vertex algebras emerge as highest weight representations of infinite-dimensional Lie algebras. The correspondence between Higgs bundles and opers, known as the non-abelian Hodge theorem (C Simpson), highlights parallels with Hodge structures. The Beilinson-Bernstein localization similarly relates variation of Hodge structures to highest weight modules over flag manifolds of semisimple Lie groups. A broader analogue of the Bernstein correspondence can also be formulated within the geometric Langlands correspondence. This framework incorporates generalized Harish-Chandra modules, known as Wakimoto modules, along with a generalized Harish-Chandra homomorphism. Finally, we conclude with an exploration of the geometric Langlands correspondence.

Explanation on the text: Section 1 is the introduction and we introduce the concept from the literature. In Section 2, we present main examples of vertex algebras we are dealing with, as affine Kac-Moody algebras and Virasoro algebras and Fock modules. We present Fock representations of Heisenberg algebra and Harish-Chandra pairs in this section. In Section 3, we give basic concepts related to variation of (mixed) Hodge structure. We explain the context of mixed Hodge modules and the non-abelian Hodge theorem of C. Simpson as equivalent notions. Section 4 contains the BeilinsonBernstein localization functor which we successively develop over the $\mathfrak{g}$-opers in order to explain the geometric Langlands correspondence. We give a brief explanation of KZ-equations and the conformal blocks at the end.

2 Vertex algebras

In this section main examples of vertex algebras and their representations are presented along what we explained in the introduction [3,4, 6-10].

Definition 2.1. A vertex algebra consists of the following data;

- (*space of states*) *A* $\mathbb{Z}$*-graded vector space*
$$V = \bigoplus_n V_n \tag{6}$$
- (*vacuum vector*) *a vector* $|0\rangle \in V_0$
- (*translation operator*) *a linear operator* $T : V \to V$ *of degree one.*
- (*vertex operators*) *a linear operation*
$$Y(.,z) : V \to End(V)[[z^{\pm 1}]] \tag{7}$$
taking $A \in V_m$ *to*
$$Y(A, z) = \sum_n A(n)\, z^{-n-1} \tag{8}$$
of conformal dimension m, i.e deg $A(n) = -n + m = 1$.
- (*vacuum axiom*) $Y(|0\rangle) = Id_V$. *Furthermore*
$$Y(A, z)\,|0\rangle \in V[[z]], \forall A \in V \tag{9}$$
- (*translation axiom*) *For any* $A \in V$,
$$[T, Y(A, z)] = \partial_z Y(A, z) \tag{10}$$
and $T|0\rangle = 0$.
- (*locality axiom*) *All fields are local with respect to each others.*

Vertex algebra structure naturally appears in many known geometric structures. Let us begin with Lie algebra H defined as central extension
$$0 \to \mathbb{C}.1 \to H \to \mathbb{C}((t)) \to 0 \tag{11}$$

It may also be regarded as the completion of the one dimensional central extension of the commutative Lie algebra of Laurent polynomials $\mathbb{C}[t, t^{-1}]$ having basis $b_n = t^n$, $n \in \mathbb{Z}$ and the central element 1. Let us call the latter Lie algebra by H'. The universal enveloping algebra $U(H')$ is an associative algebra with generators b_n and relations
$$b_n b_m - b_m b_n = n.\delta_{n,-m}1, \quad b_n.1 - 1.b_n = 0 \tag{12}$$

The left ideals $t^N\,\mathbb{C}[t]$ build up a system of open neighborhoods of 0, and one can consider the completion of $U(H')$ with respect to this topology, denoted $\widetilde{U}(H')$. The quotient
$$\widetilde{H} = \widetilde{U}(H')/(1-1) \tag{13}$$
is the well known Weyl algebra. Here the first 1 is the central element and the second is the unit of $\widetilde{U}(H')$. Let $\widetilde{H}_+$ be the subalgebra of $\widetilde{H}$ generated by b_n, $n \geq 0$ and define
$$V = Ind_{\widetilde{H}_+}^{\widetilde{H}}\,\mathbb{C} = \widetilde{H}_- = \mathbb{C}[b_{-1}, b_{-2}, \ldots] \tag{14}$$
The module V is called the Fock representation of $\widetilde{H}$. Now lets look at to the fields
$$b(z) = \sum_n b_n\, z^{-n-1} \tag{15}$$
where b_n is considered as an endomorphism of V. Since $\deg(b_n) = -n$, $b(z)$ is a field of conformal dimension one. Let us consider
$$b(z)^2 = \sum_n \left(\sum_{k+l=n}\right)(b_k\, b_l)\, z^{-n-2} \tag{16}$$

The relations (37) imply that the coefficient operators can be rearranged so that the annihilation operators (b_n, $n < 0$) be in the right side of creation operators (b_n, $n \geq 0$) and for any $x \in V$ there are only a finite number of $b_k\, b_l$ whose action on x is non zero. This makes the expression (16) well defined. There are standard ways in conformal field theory to remove infinite sums arising from repeatedly creating and annihilating the same state. In our case we define the normally ordered product of $b(z)$ with itself as

$$: b(z)\, b(z) = \sum_n : b_k\, b_l : z^{-n-2}, \qquad b_k\, b_l := \begin{cases} b_l\, b_k & l = -k, k \geq 0 \\ b_k\, b_l & \text{otherwise} \end{cases} \tag{17}$$

With the normally ordered product we can proceed to define for instance

$$Y\,(b^2_{-1}, z) =: b(z)^2 : \tag{18}$$

etc .... The pattern explained above appears in many Lie algebra representations in finite or infinite dimensions. We will encounter several examples of this in the following.

(i) *Affine Kac-Moody algebras*

The first class of vertex operator algebras corresponds to affine Lie algebras. Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra over $\mathbb{C}$. The associated loop algebra $L\mathfrak{g}$ is defined as $\mathfrak{g}((t))$. An affine Lie algebra $\hat{\mathfrak{g}}$ can be expressed as the direct sum of vector spaces $\hat{\mathfrak{g}} := L\mathfrak{g} \oplus \mathbb{C}K$, with the commutation relations given by $[K,\cdot] = 0$ and

$$[A \oplus f(t), B \oplus \mathfrak{g}(t)] = [A, B] \oplus f(t)\, \mathrm{g}(t) + (\mathrm{Res}_{t=0} f\, dg)\, (A, B)K,$$

where $(\cdot,\cdot)$ is an invariant bilinear form on $\mathfrak{g}$, normalized such that $(\theta, \theta) = 2$, with θ denoting the highest root of $\mathfrak{g}$.

A related structure is the vacuum representation of the affine algebra $\hat{\mathfrak{g}}$. For $k \in \mathbb{C}$, let $\mathbb{C}_k$ denote the 1-dimensional representation of $\mathfrak{g}[[t]] \oplus \mathbb{C}K$, where K acts as multiplication by k. The vacuum representation of $\hat{\mathfrak{g}}$ at level k is defined as

$$V_k(\mathfrak{g}) = Ind^{\hat{\mathfrak{g}}}_{\mathfrak{g}[[t]] \oplus \mathbb{C}K}\, \mathbb{C}_K.$$

If $\{J^{\,a}\}$ is a basis for $\mathfrak{g}$, then the elements $J^a_n = J_a \oplus t^n$ and K form a basis of $\hat{\mathfrak{g}}$. The vacuum representation $V_k(\mathfrak{g})$ is spanned by monomials of the form $J^{a_1}_{n_1} \ldots J^{a_m}_{n_m}\, 1_k$. This structure defines a vertex algebra (or module) $V_k(\mathfrak{g})$ with the vertex operator

$$Y\,(J^a_{-1} \cdot 1_{k,z}) = J^a(z) := \sum_n J^a_n\, z^{-n-1}.$$

The module $V_k(\mathfrak{g})$ contains a unique maximal proper $\hat{\mathfrak{g}}$-submodule, denoted by $J(k)$. The quotient

$$L_{\mathfrak{g}}(k,0) = V_k(\mathfrak{g})/J(k)$$

defines a simple vertex algebra. For a highest weight $\lambda \in h^*$ of $\mathfrak{g}$, the corresponding highest weight module for $\hat{\mathfrak{g}}$ is denoted by $L_{\mathfrak{g}}(k, \lambda)$ (see [6,7]).

(ii) *Virasoro algebras*:

The second class of vertex algebras we examine are the Virasoro algebras, denoted Vir. These algebras are a central extension of the Lie algebra $Der\, \mathbb{C}((t))$, generated by the operators $L_n = -\,t^{n+1} \dfrac{d}{dt}$ for $n \in \mathbb{Z}$, along with the 1-dimensional vector space $\mathbb{C} \cdot C$, subject to the relations $[C,\cdot] = 0$ and

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3 - n}{12}\, \delta_{n,-m}\, C.$$

Let $c,h \in C$, and define the 1-dimensional representation $\mathbb{C}_c$ of Vir by the following actions:

$$L_n \cdot 1 = 0 \qquad \text{for } n \geq 1,$$

$$L_0 \cdot 1 = h \cdot 1,$$

$$C \cdot 1 = c \cdot 1.$$

Next, we define the module $V(c,h)$ as

$$V(c,h) := Ind_{Der\mathbb{C}[[t]] \oplus C}^{\mathrm{Vir}} \mathbb{C}_c,\ c \in \mathbb{C}.$$

The module $V(c,h)$ is generated by the monomials $L_{j1} \cdots L_{jm} 1_c$ with $j_1 \leq j_2 \leq \cdots \leq -2$, and it forms a vertex algebra module with the vertex operator given by

$$Y(L_{-2}\, 1_c, z) = T(z) = \sum_n L_n\, z^{-n-2}$$

Modules over the Virasoro algebra are classified based on the action of the operator $L(0)$, where $L(0) \cdot 1 = h \cdot 1$. Such modules are highest weight modules of the Virasoro algebra, denoted $V(c, h)$. The unique irreducible quotient of $V(c, h)$ is denoted by $L(c, h)$, as described in [6,10].

(*iii*) (*Bosonic*) *Fock representations:*

Fock modules are defined using the Heisenberg Lie algebra of rank 1, given by

$$H = \bigoplus_n \mathbb{C}a_n \oplus \mathbb{C}K, \quad [K, \cdot] = 0, \quad [a_m, a_n] = m\delta_{m+n,0}K.$$

Let $\mathbb{C}_\eta$ be the 1-dimensional $H^{\geq} = (H^0 \oplus H^+)$-representation with the following actions:

$$a_n \cdot 1_\eta = \eta\, \delta_{n,0}\, 1_\eta,\ K \cdot 1_\eta = 1_\eta.$$

This representation has a $\mathbb{Z}$-gradation, where

$$\mathbb{C}_\eta^n = \begin{cases} \mathbb{C}_n & \text{for} \quad n = 0, \\ 0 & \text{for} \quad n \neq 0 \end{cases}$$

The corresponding (bosonic) Fock module is defined by

$$\mathcal{F}^\eta = Ind_{H\geq}^{H}\, \mathbb{C}_\eta.$$

The highest weight vector in this module is denoted by $|\eta\rangle$. We define a $\mathbb{Z}$-graded vertex algebra structure on $\mathcal{F}^0$ with vacuum vector $|0\rangle$ and translation operator

$$T|0\rangle = 0,\ [T, a_n] = -\, na_{-n-1}.$$

The vertex operator is given by

$$Y(a_{-1}\, |0\rangle, z) = \sum_n a_n\, z^{-n-1}.$$

Fock representations can be understood as the smallest representations of the Weyl algebra, as discussed in [7,10].

(*iv*) *Harish-Chandra modules:*

A pair $(\mathfrak{g}, K)$, where $\mathfrak{g}$ is a Lie algebra and K is a Lie group such that $\mathfrak{k} = \mathrm{Lie}(K)$, together with an action

$$Ad : K \longrightarrow \mathfrak{g},$$

is called a Harish-Chandra pair if it is compatible with the adjoint action of K on $\mathfrak{k}$. A $(\mathfrak{g}, K)$-action on a scheme X consists of a homomorphism

$$\rho : \mathfrak{g} \rightarrow \Theta_X,$$

where Θ_X is the tangent sheaf of X, along with an action of K on X, such that the following conditions hold:

(1) The differential of the K-action is the restriction of the action of $\mathfrak{g}$ on $\mathfrak{k}$.

(2) $\rho\big(Ad(k)(a)\big) = k\rho(a)k^{-1}$, for all $k \in K$ and $a \in \mathfrak{g}$.

A Harish-Chandra $(\mathfrak{g}, K)$-module is a vector space V equipped with the aforementioned compatible actions. One can also consider the vector bundle

$$\mathcal{V} = X \times_K V$$

on the scheme X, which provides a flat connection on the trivial vector bundle $X \times V$ over X.

Harish-Chandra modules can be generalized to Virasoro algebras. In this context, the Lie algebra $\mathfrak{g}$ admits a generalized triangular decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha,$$

where $\mathfrak{h}$ is a Cartan subalgebra. In this case, the $\mathfrak{g}$-module M is assumed to be h-diagonalizable, i.e.,

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda, \qquad \dim M_\lambda < \infty,$$

where each weight space M_λ is finite-dimensional. The module M can then be written as a direct sum of highest weight modules, lowest weight modules (Verma modules), or an intermediate series defined by $V_{a,b}$, where $a,b \in \mathbb{C}$. The intermediate series is defined by

$$V_{a,b} = \bigoplus_n \mathbb{C}v_n, \quad L_s \cdot v_n = (as + b - n)v_{n+s}, \quad C \cdot v_n = 0,$$

where L_s are the generators of the Virasoro algebra. For further details, see [7] and [10].

Jantzen filtration and Shapovalov form: Let $(\mathfrak{g}, \mathfrak{h})$ be a Lie algebra pair, where $\mathfrak{h}$ is a Cartan subalgebra, and $\sigma : \mathfrak{g} \to \mathfrak{g}$ is an anti-involution. The framework discussed here is applicable to any Q-graded Lie algebra:

$$\mathfrak{g} = \bigoplus_{\beta \in Q} \mathfrak{g}_\beta, \qquad \dim \mathfrak{g}_\beta < \infty,$$

where Q is an abelian group, and in our case, Q is the root lattice. This definition is relevant for both finite-dimensional Lie algebras and their infinite-dimensional extensions, such as affine or Virasoro algebras.

We can decompose $\mathfrak{g}$ as:

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+.$$

The universal enveloping algebra of $\mathfrak{g}$ is defined as the quotient of the tensor algebra $T(\mathfrak{g}) = \oplus_n \mathfrak{g}^{\otimes n}$ by the ideal generated by $x \otimes y - y \otimes x - [x, y]$. According to the Poincaré-Birkhoff-Witt theorem, the universal enveloping algebra has the decomposition:

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus \{\mathfrak{g}_- U(\mathfrak{g}) + U(\mathfrak{g})\, \mathfrak{g}_+\}.$$

Consider the projection:

$$\pi : U(\mathfrak{g}) \to U(\mathfrak{h}) \cong S(\mathfrak{h}),$$

where $S(\mathfrak{h})$ is the symmetric algebra of $\mathfrak{h}$. The bilinear form

$$F : U(\mathfrak{g}) \times U(\mathfrak{g}) \to S(\mathfrak{h}), \quad F(x, y) = \pi(\sigma(x)\, y),$$

is referred to as the Shapovalov form of $\mathfrak{g}$. It is a symmetric, contravariant form, meaning:

$$F(zx, y) = F(x, \sigma(y)),$$

for all $x, y \in U(\mathfrak{g})$. The decomposition of $U(\mathfrak{g})$ implies a corresponding decomposition of the algebra:

$$U(\mathfrak{g}) = \bigoplus_\beta U(\mathfrak{g})_\beta\,, U(\mathfrak{g})_\beta = \{x \in U(\mathfrak{g}) | [h, x] = \beta(h)x,\ \forall h \in \mathfrak{h}\}.$$

For distinct $\beta_1 \neq \beta_2 \in Q$, it follows that:

$$F(x, y) = 0, \qquad x \in U(\mathfrak{g})_{\beta_1}, \qquad y \in U(\mathfrak{g})_{\beta_2}.$$

For each $\beta \in Q$, a basis $\{X_j\}_{j \in I}$ of $U(\mathfrak{g})_{-\beta}$ can be chosen. The determinant

$$D_\beta = \det\big(F(X_i, X_j)\big)_{i, j \in I} \in S(\mathfrak{h}),$$

is called the Shapovalov determinant of $\mathfrak{g}$.

The concept of a contravariant bilinear form can be extended to any $\mathfrak{g}$-module M. A key property in this context is that any highest weight module M has a unique contravariant bilinear form $\langle\cdot,\cdot\rangle : M \otimes M \to \mathbb{C}$, up to a constant, satisfying the condition

$$\langle \mathfrak{g} \cdot x, y\rangle = \langle x, \sigma(\mathfrak{g}) \cdot y\rangle, \qquad g \in U(\mathfrak{g}).$$

This fact can be verified for Verma modules $M(\lambda)$, which are the unique irreducible quotients of these modules. For a Verma module M_λ, the radical of the form is the maximal proper submodule $J(\lambda) \subset M(\lambda)$.

The Jantzen filtration of a $\mathfrak{g}$-module is defined using the Shapovalov form on $U(\mathfrak{g})$. Let $R = \mathbb{C}[t]$ and $\phi : R \to \mathbb{C}$ be the canonical map.

Let $\tilde{M}$ be a free R-module of rank r with a non-degenerate symmetric bilinear form

$$(.,.)_{\tilde{M}} : \tilde{M} \times \tilde{M} \to R.$$

Define $M = \phi\, \tilde{M} = \tilde{M} \otimes_R R/tR$, and the symmetric bilinear form on M is given by:

$$(\phi\, v_1, \phi v_2) = \phi((v_1, v_2)_{\tilde{M}}).$$

For $m \in \mathbb{Z}_{\geq 0}$, define:

$$\tilde{M}(m) = \{v \in \tilde{M} \mid (v, \tilde{M})_{\tilde{M}} \subset t^m R\},$$

and embed $\tilde{M}(m) \to \tilde{M}$. Set $M(m) = \mathrm{Im}(\phi \text{ o } i_m)$, and we obtain a filtration:

$$M = M(0) \supset M(1) \supset \cdots,$$

which is the Jantzen filtration of M, a filtration of $\mathbb{C}$-vector spaces. This filtration has the following properties:

- $\cap_m M(m) = 0$,
- $M(1) = \mathrm{rad}\,(\cdot,\cdot)$,
- There exists a symmetric bilinear form $(.,.)_m$ on $M(m)$ such that $\mathrm{rad}((.,.)_m) = M(m+1)$.

The concept of the Jantzen filtration appears both in the context of vertex algebras and in variations of Hodge structures, where it corresponds to the weight filtration in local systems of mixed Hodge structures.

(v) Conformal vertex algebras:

A vertex algebra $V = \oplus_n V_n$ with central charge c is called conformal if it contains a vector $w \in V_2$ (known as the conformal vector) such that the corresponding vertex operator $Y(w,z) = \sum_n L_n z^{-n-2}$ satisfies the following conditions:

$$L_{-1} = T,\ L_0|V_n = n \cdot Id,\ L_2 w = 1/2\ c|0\rangle.$$

This implies that there is a homomorphism

$$V ir_c \to V, \qquad L_{-2} 1_c \to w.$$

In the case of a Kac-Moody algebra, the conformal vector (called the Sugawara conformal vector) is given by

$$\frac{1}{2(k \times h^v)} \sum_a (J_{-1}^a)^2\, 1_k,$$

where J^a is an orthonormal basis of $\mathfrak{g}$. A Kac-Moody algebra is conformal if and only if $k \neq -h^\vee$. In this case, $\hat{\mathfrak{g}}$ is a module over the Virasoro algebra [7,11].

Unitary Vertex Algebras:

Let $(V, Y, 1, w)$ be a vertex algebra, and let $\phi : V \to V$ be an antilinear involution. The pair (V, ϕ) is called unitary if there exists a positive definite Hermitian form

$$(.,.) : V \times V \to \mathbb{C}$$

such that for $a, u, v \in V$,

$$(Y(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})\, u, v) = (u, Y(\phi(a), z)\, v),$$

where

$$\mathrm{Y}(w, z) = \sum_n L(n)\, z^{-n-2}$$

In a unitary vertex operator algebra, the positive definite Hermitian form is uniquely determined by $(1,1)$ via the properties of the vertex algebra. For the Virasoro case $V(c, h)$, there exists a unique Hermitian form such that

$$(1_{c,h}, 1_{c,h}) = 1, \qquad (L_n u, v) = (u, L_{-n} v).$$

It is known that $V(c,h)$ is unitary if and only if $c \geq 1$, $h \geq 0$, or $c = c_m$, $h = h^m_{r,s}$,

where

$$c_m = 1 - \frac{6}{m(m+1)}, \qquad h^m_{r,s} = \frac{[r(m+1) - sm]^2 - 1}{4m(m+1)}.$$

In the affine case $V_{\mathfrak{g}}(k, \lambda)$, there exists a unique positive definite Hermitian form such that

$$(1, 1) = 1, (xu, v) = -(u, \hat{\omega}_0(x)v), \mathrm{x} \in \hat{\mathfrak{g}}, u, v \in L_{\mathfrak{g}}(k, \lambda),$$

where $\hat{\omega}_0 : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$

is the Cartan involution. Then, $V_{\mathfrak{g}}(k, \lambda)$ is a unitary vertex algebra if and only if $k \neq -h^\vee$, $k \in \mathbb{Z}^+$, and λ is a dominant integral weight satisfying $(\lambda, \theta) \leq k$.

In the Fock module case, we reduce to $M(1, \lambda) = U(\hat{\mathfrak{h}})/J_\lambda$. It is known that this is unitary if and only if $(\alpha, \lambda) \geq 0$, i.e., λ is a dominant weight [6].

3 Variation of Hodge structure

A variation of Hodge structure [12-17] over a complex manifold S gives rise to a period map

$$\Phi : S \to \Gamma_{\mathbb{Z}} / D$$

where S is a smooth base manifold and Γ is a discrete group. D is the period domain and it is known that it is a hermitian symmetric complex manifold. There are naturally defined Hodge bundles F^p of the Hodge structure on V, and also the endomorphism bundle associated to $\mathfrak{g} = End(V)$ on D. The corresponding local systems are

$$V := \Gamma\backslash(D \times V), \quad G := \Gamma\backslash(D \times \mathfrak{g}) \tag{19}$$

respectively. One way to explain the complex structure on D is to embed it in its compact dual $\check{D}$, which is the set of all Hodge filtrations on V with the same Hodge numbers satisfying the first Riemann-Hodge bilinear relation. $\check{D}$ is a homogeneous complex manifold. There are $G_{\mathbb{C}}$-homogeneous vector bundles

$$F^p \to \check{D} \tag{20}$$

called Hodge bundles whose fiber at a given point $F^\bullet$ is F^p. Over $D \subset \check{D}$ we have $V^{p,q} = F^p/F^{p+1}$, which are homogeneous vector bundles for the action of $G_{\mathbb{R}}$. They are Hemitian vector bundles with $G_{\mathbb{R}}$-invariant Hermitian metric given in each fiber by the polarization form. The space of functions on D can be identified with the $\Gamma_{\mathbb{Z}}$-automorphic functions on D.

(i) *Variation of mixed Hodge structure:*

A polarized variation of mixed Hodge structure over the punctured disc Δ^* consists of a 5-tuple

$(\mathbf{V}, F^{\bullet}, W_{\bullet}, \nabla, P)$, where:

- $\mathbf{V}$ is a local system of $\mathbb{Q}$-vector spaces on Δ^*.
- $W_{\bullet}$ is an increasing filtration on V by sub-local systems of $\mathbb{Q}$-vector spaces.
- $F^{\bullet} = (F^i)$ is a decreasing filtration on the vector bundle $V \otimes_{\mathbb{Q}} \mathcal{O}_{\Delta^*}$ by holomorphic sub-bundles.
- $\nabla : \mathcal{V} \otimes_{\mathbb{Q}} \mathcal{O}_{\Delta^*} \to \mathcal{V} \otimes_{\mathbb{Q}} \Omega^1_{\Delta^*}$ is a flat connection satisfying Griffiths transversality: $\nabla(F^i) \subset F^{i-1} \otimes \Omega^1_{\Delta^*}$.
- $P : \mathbf{V} \times \mathbf{V} \to \mathbb{Q}$ is a flat pairing inducing a set of rational, nondegenerate bilinear forms $P_k : Gr_W^k \mathbf{V} \otimes Gr_W^k \mathbf{V} \to \mathbb{Q}$, such that the triple $(Gr_W^k \mathbf{V}, F^{\bullet} Gr_W^k, P_k)$

defines a pure polarized variation of Hodge structure on Δ^*. We denote this structure as $\mathcal{H}$.

Next, let V be a Hodge structure with an exhaustive decreasing Hodge filtration F^p. Consider a locally free sheaf $\xi(V,F)$ over $\mathbb{C}$, defined as the submodule of $V \otimes \mathbb{C}[t, t^{-1}]$ generated by $t^{-p}F^p$. Given a real structure, $\xi(V, F)$ and $\xi(V,\bar{F})$ can be glued using the involution $t \to (\bar{t}\,)^{-1}$, yielding a locally free sheaf $\xi(V,F,\bar{F})$ on $\mathbb{P}^1$ with an action of $\mathbb{C}^*$ and an antilinear involution. This procedure can be described as follows. A variation of polarized Hodge structure of weight k provides a 4-tuple (H, F, ∇, P), where:

$$\nabla : H \to H \otimes z^{-1}\Omega_{\Delta^*}(\log 0)$$

is a flat connection, and a $(-1)^k$-symmetric, non-degenerate, flat pairing

$$P : H \times j^* H \to \mathcal{O}_{\Delta^*}, \qquad j : z \to -z$$

is induced. The bilinear form P induces a non-degenerate symmetric pairing

$$z^{-k}P : H/zH \times H/zH \to \mathbb{C}.$$

The Hodge filtration can be described as follows. Let V be the Kashiwara Malgrange filtration on the mixed Hodge module associated with H, and assume (H,∇) is regular singular. Then $H \hookrightarrow V > -\infty$. For $\alpha \in [0,1]$, define (see [12]):

$$F^p H_\lambda := z^{\beta + \frac{N}{2\pi i}} Gr_V^{\beta}\ H.$$

We will identify the variation of polarized Hodge structures with their associated polarizable Hodge module via the Riemann-Hilbert correspondence. This correspondence has been studied more systematically by Saito in [16].

(ii) *Polarizable Mixed Hodge modules:*

Let X be a complex algebraic variety, and let $MHM(X)$ denote the abelian category of Mixed Hodge Modules on X. $MHM(X)$ comes equipped with a forgetful functor

$$\mathrm{rat} : MHM(X) \to \mathrm{Perv}(\mathbb{Q}X),$$

which assigns to each mixed Hodge module its underlying perverse sheaf over $\mathbb{Q}$. Sometimes, these objects are considered as elements in $D^bMHM(X)$ and $D_c^b(\mathbb{Q}X)$, respectively, and similarly for the functor rat. When X is smooth, a mixed Hodge module on X is characterized by a 4-tuple (M,F,K,W), where M is a holonomic D-module with a good filtration F, and it has a rational structure such that,

$$\mathrm{DR}(M) \cong \mathbb{C} \otimes K \in \mathrm{Perv}(\mathbb{C}_X),$$

for a perverse sheaf K and W denote the pair of weight filtrations on M and K that are compatible with the rat functor. Here, DR refers to the de Rham functor, shifted by dim (X). The de Rham functor is dual to the solution functor. When $X = pt$, $MHM(pt)$ precisely corresponds to all the polarizable mixed Hodge structures.

A mixed Hodge module always has a weight filtration W, and it is called pure of weight n if $Gr^k_W = 0$ for $k \neq n$. Typically, the weight filtration W involves a nilpotent operator on M or the underlying variation of mixed Hodge structure. A mixed Hodge module is constructed by successive extensions of pure ones. If the support of a pure Hodge module, considered as a sheaf, is irreducible and no submodule or quotient module has a smaller support, we say the module has strict support. Any pure Hodge module decomposes uniquely into pure modules with different strict supports, as guaranteed by the Decomposition Theorem. A pure Hodge module is also referred to as a polarizable HM. We denote by $MH_Z(X,n)^p$ the category of pure Hodge modules with strict support Z.

An element $M \in HM_Z(X,n)$ determines a polarizable variation of Hodge structure. The converse is also true: variations of Hodge structures determine mixed Hodge modules. Therefore,

$MH_Z(X,n)^p \simeq V\,HS_{gen}(Z,n - \dim Z)^p$, where the right-hand side refers to polarizable variations of Hodge structure of weight $n - \dim Z$ defined on a non-empty smooth subvariety of Z. This equation reflects a profound and non-trivial result about regular holonomic D-modules, their underlying perverse sheaves, and their polarizations. It may also be interpreted as an analogue of the Riemann-Hilbert correspondence between mixed Hodge modules and their underlying perverse sheaves, as discussed in [14].

(*iii*) *Higgs Bundles and the Non-Abelian Hodge Theorem:*

Let X be a smooth and projective variety over $\mathbb{C}$. A harmonic bundle on X is a C^∞-vector bundle E with differential operators ∂ and $\bar{\partial}$, and algebraic operators θ and $\bar{\theta}$ satisfying the following conditions: There exists a metric h such that $\partial + \bar{\partial}$ is a unitary connection, and $\theta + \bar{\theta}$ is self-adjoint. Furthermore, if

$$\nabla = \partial + \bar{\partial} + \theta + \bar{\theta}, \qquad \nabla'' = \bar{\partial} + \theta,$$

then we have $\nabla^2 = \nabla''^2 = 0$. Under these conditions, (E,D) is a vector bundle with a flat connection, and (E,∂,θ) is a Higgs bundle, i.e., a holomorphic vector bundle with a holomorphic section

$$\theta \in H^0(\mathrm{End}(E) \otimes \Omega^1_X), \quad \theta \wedge \theta = 0.$$

A Higgs bundle is called stable (resp. semistable) if for any coherent subsheaf $F \subset E$ preserved by θ, the inequality

$$\frac{\deg(F)}{\mathrm{rank}(F)} < \frac{\deg(E)}{\mathrm{rank}(E)} \text{ (resp. } \leq)$$

holds.

There is a natural equivalence between the categories of harmonic bundles on X and semisimple flat bundles (or representations of $\pi_1(X)$). There is also a natural equivalence between the categories of harmonic bundles and direct sums of stable Higgs bundles with vanishing Chern class. This correspondence between representations and Higgs bundles extends to an equivalence between the category of all representations of $\pi_1(X)$ and all semistable Higgs bundles with vanishing Chern classes, referred to as the non-abelian Hodge theorem.

A natural $\mathbb{C}^*$-action exists on the category of semistable Higgs bundles with vanishing Chern classes, denoted by

$$t : (E,\theta) \to (E,t\theta).$$

The semistable representations fixed by this action correspond exactly to complex variations of Hodge structure. A representation ϱ of $\pi_1(X)$ is called rigid if any nearby representation is conjugate to it. The correspondence described above is continuous on the moduli of semisimple representations. It follows that if a semisimple representation is rigid, it must be fixed by $\mathbb{C}^*$ and originates from a complex variation of Hodge structure. In this case, there is a $\mathbb{Q}$-variation of Hodge structure $V_{\mathbb{Q}}$ such that ϱ is a direct factor of the monodromy representation of $V_{\mathbb{Q}} \otimes \bar{\mathbb{Q}}$ (where the monodromy is the sum of conjugates of ϱ) [15].

Let $M_{Dol}(G), M_{DR}(G), M_B(G)$ denote the moduli spaces of Higgs bundles of degree zero, local systems, and representations of $\pi_1(X)$, respectively. We will denote the smooth loci of these varieties by the superscript reg, i.e., $M_{Dol}^{reg},\ldots$, and typically omit the "reg" notation in the future. The non-abelian Hodge theorem provides a diffeomorphism

$$\tau : M_{Dol}(G) \cong M_{DR}(G).$$

The Riemann-Hilbert correspondence between bundles with integrable connections and representations yields isomorphisms

$$M_{DR}(G) \cong M_B(G),$$

as discussed in [15,18]. A systematic study of the interrelation between the Higgs fields of Higgs bundles and the system of Hodge bundles in variations of Hodge structures can be found in [13]. A corollary of this is that a unipotent variation of mixed Hodge structure defines a Higgs field θ which is flat with respect to ∇ and $\partial + \bar{\partial}$. In this case, the invariance under the C^*-action described above explains the complex variation of the Hodge filtration.

4 Connection between Hodge structure and Vertex algebras

Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C} = Lie(G_{\mathbb{R}} \otimes \mathbb{C})$ be a complex semi-simple Lie algebra, $\mathfrak{h} = t \otimes \mathbb{C}$ a Cartan subalgebra, $t = Lie(T)$, and $K_{\mathbb{C}}$ a complex Lie group corresponding to the unique maximal compact subgroup $K \subset G_{\mathbb{R}}$. We denote $U(\mathfrak{g}_{\mathbb{C}})$ to be the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. We assume the action of $K_{\mathbb{C}}$ will be locally finite, and its differential agrees with the corresponding subspace of $U(\mathfrak{g}_{\mathbb{C}})$. One may match these data with the case $D = G_{\mathbb{R}}/H$ is a general period domain sitting in the diagram

$$\begin{tikzcd} G_{\mathbb{R}}/T \arrow[r] \arrow[d] & G_{\mathbb{C}}/B \arrow[d] \\ D = G_{\mathbb{R}}/H \arrow[r] & G_{\mathbb{C}}/P = \check{D} \end{tikzcd} \tag{21}$$

with with T a maximal torus, B a Borel subgroup, and horizontal arrows to be inclusions. Let $U(\mathfrak{h}_{\mathbb{C}})$ be the universal enveloping algebra of $\mathfrak{h}$. The Weyl group W of $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{C}})$ acts on $U(\mathfrak{h}_{\mathbb{C}})$ and gives an isomorphism

$$HC : Z(\mathfrak{g}_{\mathbb{C}}) \xrightarrow{\cong} U(\mathfrak{h}_{\mathbb{C}})^w \tag{22}$$

where the upper-index means the elements fixed by W. Using the isomorphism (22) one can assign to each positive root $\mu \in \mathfrak{h}_{\mathbb{C}}^*$ the homomorphism

$$\chi_\mu : Z(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}, \quad z \to HC(z)(\mu) \tag{23}$$

is called the infinitesimal character associated to μ. A result of Harish-Chandra says that any character of $Z(\mathfrak{g}_{\mathbb{C}})$ is an infinitesimal character, and

$$\chi_\mu = \chi_{\mu}' \iff \mu = w(\mu'),\ w \in W \tag{24}$$

We will use the above set up in our construction of correspondence between Hodge bundles and vertex algebras. In fact our correspondence uses a generalization of Eq (22). We will do this step by step as follows.

(*i*) *Beilinson-Bernstein correspondence:*

Let G be a connected complex reductive algebraic group defined over $\mathbb{R}$, and let $\mathfrak{b} = \mathfrak{b} \oplus n \subset \mathfrak{g} =$ Lie(G) be the Borel subalgebra, where the unipotent radical is given by

$$n = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}^{-\alpha}.$$

Here, $\mathfrak{h}$ is the Cartan subalgebra. Any element λ in the weight lattice $\Lambda \subset \mathfrak{h}_{\mathbb{R}}^*$ lifts to an algebraic character e^λ of the Borel subgroup $B \subset G$, corresponding to $\mathfrak{b}$. To this character, there exists a unique

G-equivariant line bundle $\mathcal{L} \to X = G/B$, with the group B acting as e^{λ} on the fibers. Thus, Λ can be interpreted as the set of G-equivariant line bundles via $\lambda \to \mathcal{L}_{\lambda}$.

Let $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha \in \frac{1}{2}\Lambda$. Then,

$$\mathcal{L}_{-2\rho} \cong \bigwedge^n T^*X.$$

If G is simply connected, then $\rho \in \Lambda$, and $\mathcal{L}_{-2\rho}$ has a well-defined square root. Now, define the sheaf

$$D_{\lambda} = \mathcal{O}(\mathcal{L}_{\lambda-\rho}) \otimes D_X \otimes \mathcal{O}(\mathcal{L}_{\rho-\lambda}).$$

The Lie algebra $\mathfrak{g}$ acts by infinitesimal translations on sections of $\mathcal{L}_{\lambda-\rho}$, so $\mathfrak{g} \hookrightarrow \Gamma D_{\lambda}$, which induces a map

$$U(\mathfrak{g}) \to \Gamma D_{\lambda}.$$

The center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ acts via the infinitesimal character χ_{λ}. This gives a homomorphism

$$U_{\lambda}(\mathfrak{g}) = U(\mathfrak{g})/\ker(\chi_{\lambda}) \to \Gamma D_{\lambda}.$$

This map is compatible with the degree filtration. The Beilinson-Bernstein theorem asserts that the map above is an isomorphism. Let $\mathrm{Mod}(U_{\lambda}(\mathfrak{g}))_{\mathrm{fg}}$ denote the category of finitely generated $U_{\lambda}(\mathfrak{g})$-modules, or equivalently, the category of finitely generated $U_{\lambda}(\mathfrak{g})$-modules on which $Z(\mathfrak{g})$ acts via χ_{λ}. Similarly, let $\mathrm{Mod}(D_{\lambda})_{\mathrm{coh}}$ refer to the category of coherent D_{λ}-modules. Define the functors:

$$\Delta : \mathrm{Mod}\,(U_{\lambda}(\mathfrak{g}))_{\mathrm{fg}} \to \mathrm{Mod}\,(D_{\lambda})_{\mathrm{coh}}, \quad \Delta(M) = M \otimes_{U_{\lambda}(\mathfrak{g})} D_{\lambda},$$

$$\Gamma : \mathrm{Mod}\,(D_{\lambda})_{\mathrm{coh}} \to \mathrm{Mod}\,(U_{\lambda}(\mathfrak{g}))_{\mathrm{fg}}, \quad \Gamma(M) = H^0(X, M) \tag{25}$$

The Beilinson-Bernstein theorem asserts that when λ is a regular and integrally dominant weight, these functors define an equivalence of categories:

$$\mathrm{Mod}(U_{\lambda}(\mathfrak{g}))_{\mathrm{fg}} \cong \mathrm{Mod}(D_{\lambda})_{\mathrm{coh}}.$$

Similarly, for λ, a corresponding isomorphism holds between the category of Harish-Chandra modules:

$$HC(\mathfrak{g}, K)_{\lambda} \cong HC(D_{\lambda}, K),$$

where the right-hand side denotes the category of D_{λ}-modules with a compatible K-action.

If we consider the polarization of D-modules as a Hermitian duality, we have the pairing

$$P : M \times \overline{M} \to C^{\infty}(X_{\mathbb{R}}), \qquad \text{bilinear over } D_{\lambda} \times \overline{D}_{-\lambda}.$$

In our context, the map $M \to \overline{M}$ defines a bijection

$$\mathrm{Mod}(D_{X,\lambda})\ \mathrm{rh} \cong \mathrm{Mod}(D_{\overline{X},-\lambda})_{\mathrm{rh}}.$$

We sometimes omit the subscripts X or $\overline{X}$ and simply write D_{λ} and $D_{-\lambda}$. The pairing P should be interpreted as $(\sigma, \tau) = \int_X \langle\sigma, \tau\rangle$, where $\langle\cdot,\cdot\rangle$ is the flat Hermitian pairing on the underlying vector bundles, which is invariant under $\mathfrak{u}_{\mathbb{R}}$, the compact form of $\mathfrak{g}$ defined by a Cartan involution. In the correspondence, the flat bilinear form P corresponds to the Shapovalov form, and the weight filtration corresponds to the Jantzen filtration [17].

(*ii*) *Localization Functor:*

We generalize the Beilinson-Bernstein correspondence. Let G be a connected simple Lie group over $\mathbb{C}$. Assume $LG = G((t))$ is the Lie group of $L\mathfrak{g} = \mathfrak{g}((t))$. Let X be a smooth projective algebraic curve over $\mathbb{C}$, and $p \in X$ be a point. Let P be a principal G-bundle over X. Define the associated vector bundle

$$\mathfrak{g}_P = P \times_G \mathfrak{g}.$$

Let $\mathfrak{g}^P_{\mathrm{out}}$ be the Lie algebra of sections of $\mathfrak{g}_P$ around $p \in X$, and G_{out} be the Lie group of $\mathfrak{g}_{\mathrm{out}}$. There is a natural embedding

$$\mathfrak{g}^P_{\mathrm{out}} \to L_{\mathfrak{g}}.$$

This embedding can be lifted to $\mathfrak{g}_{\text{out}}^{P} \to \hat{\mathfrak{g}}$. Let $\mathcal{O}^0$ be the category of $\hat{\mathfrak{g}}$-modules where the Lie subalgebra $\mathfrak{g}_{\text{in}} = \mathfrak{g}[[t]]$ acts locally finitely. The modular functor assigns to a module

$$M \to M/\mathfrak{g}_{\text{out}}^{P} M.$$

The dual space of $M/\mathfrak{g}_{\text{out}}^{P} M$ is called the space of conformal blocks. The localization functor of Beilinson and Bernstein assigns to M a D-module on the homogeneous space

$$\mathcal{M} = LG/G_{\text{out}}.$$

For any integer k, define a line bundle $\mathcal{L}^k$ on $\mathcal{M}$, together with a homomorphism from $\hat{\mathfrak{g}}$ to the Lie algebra of infinitesimal symmetries of $\mathcal{L}^k$. This gives a homomorphism from $U_k(\hat{\mathfrak{g}})$ to the algebra D_k of global differential operators on $\mathcal{L}^k$. Thus, for any $\hat{\mathfrak{g}}$-module M of level k, we can define a left D_k-module

$$\Delta(M) = D_k \otimes_{U(\hat{\mathfrak{g}})} M.$$

The fiber of $\Delta(M)$ at P is $M/\mathfrak{g}_{\text{out}}^{P} M$ [11]. The map Δ here is an analogue of the map in (22), generalizing it over affine algebras. The adjoint functor is the global section functor. As in the previous section, the Shapovalov form of M corresponds to the flat Hermitian pairing on the right-hand side. The map

$$\Delta : \hat{\mathfrak{g}}\text{-Mod} \to D_{\mathcal{M}}\text{-Mod}$$

is a generalization of the map (25).

(iii) $\mathfrak{g}$-*opers on the punctured disc:*

We begin by defining $GL(n)$-opers, which are pairs $(F_\bullet \subset E, D)$, where E is a rank n vector bundle equipped with:

- A complete flag $(0) \subset F_1 \subset \cdots \subset F_n = E$ such that $\text{rank}(F_i) = i$,
- A holomorphic connection $D : E \to E \otimes K$ that is necessarily flat, satisfying:
- Griffiths transversality: $D : F_i \to F_{i+1} \otimes K$,
- Non-degeneracy (strictness): $Gr_i D : Gr_i^F E \cong Gr_{i+1}^F E \otimes K$.

An $SL(n)$-oper is a $GL(n)$-oper where $\det(E) = \mathcal{O}$ and D induces the trivial connection on $\det(E)$.

Now, let G be a simple complex group with $B \subset G$ as a fixed Borel subgroup, $N = [B,B]$ its unipotent radical, $H = B/N$ the Cartan subgroup, and $Z = Z_G$ the center. The corresponding Lie algebras are $\mathfrak{n} \subset \mathfrak{b} \subset \mathfrak{g}$, $\mathfrak{t} = \mathfrak{b}/\mathfrak{n}$. If P_B is a holomorphic principal B-bundle, P_G the induced G-bundle, and $Conn(P_G)$ the sheaf of connections, we define the projection

$$c : Conn(P_G) \to \mathfrak{g}/\mathfrak{b}_P \otimes K,$$

with the conditions $c^{-1}(0) = Conn(P_B)$ and $c(D + \nu) = c(D) + [\nu]$, where ν is a section of $\mathfrak{g}_P \otimes K$ and $[\nu]$ is its image in $\mathfrak{g}/\mathfrak{b}_P \otimes K$. We consider the class

$$c(D) \in H^0(\mathfrak{g}/\mathfrak{b}_P \otimes K),$$

obtained by taking locally any flat B-connection D_B, and then gluing the local sections $[D - D_B]$. A G-oper on X is a pair (P_B, D), where $D \in H^0(Conn(P_G))$ such that

- $c(D) \in H^0((\mathfrak{g}_{-1})_P \otimes K) \subset H^0(\mathfrak{g}/\mathfrak{b}_P \otimes K)$, - For every simple negative root α, the component $c(D)_\alpha \in H^0(\mathfrak{g}/\mathfrak{b}_P \otimes K)$ is nowhere vanishing.

These conditions ensure that the connection D preserves the flag corresponding to the Borel subgroup B via Griffiths transversality. By definition, a $\mathfrak{g}$-oper is a G-oper where G is the group of inner automorphisms of $\mathfrak{g}$.

For $GL(n)$, the oper condition implies that if $E_U \cong \mathcal{O}^n$ is a trivialization compatible with the flag on an open chart U, then the flat connection can be written as

$$d+\begin{bmatrix} * & * & \cdots & * & * \\ \times & * & \cdots & * & * \\ 0 & \times & \cdots & * & * \\ \vdots & \vdots & \vdots & \vdots & * \\ 0 & 0 & \cdots & \times & * \end{bmatrix} dt$$

Thus, a B-gauge equivalence class of a GL_n-oper has a unique representative of the form

$$d+\begin{bmatrix} a_1 & a_2 & \cdots & \cdots & a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} dt.$$

When we discuss $\mathfrak{g}$-opers, we are automatically considering their Bgauge equivalence classes, [18]. Let's restrict to opers on the punctured disc. By definition, the space of $\mathfrak{g}$-opers on the punctured disc $D^{\times}$ is

$$\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(D^{\times}) = \left\{\sum_i \psi_i X_{-\alpha i} + v \mid \psi_i \neq 0 \in \mathbb{C}((t)),\ v \in \mathfrak{b}((t))\right\} \Big/ B((t)),$$

where α_i are the set of positive simple roots of $\mathfrak{g}$ with respect to B. The action of $B((t))$ is via the gauge transformation

$$g \cdot D = Ad(g)D - (\partial_t g)\, g^{-1}.$$

An oper on the punctured disc is called nilpotent if its connection has a regular singularity at the origin with unipotent monodromy. $\mathfrak{g}$-opers, as gauge equivalence classes of flat connections, can be compared with mixed Hodge modules [17,19,20,10,18,21].

(*iv*) $\hat{\mathfrak{g}}$*-modules associated to opers*: Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra and $\hat{\mathfrak{g}}$ its affine Lie algebra. Define the following:

$$\tilde{U}(\hat{\mathfrak{g}}) := \lim_{\leftarrow} U(\hat{\mathfrak{g}})/U(\hat{\mathfrak{g}})(\mathfrak{g} \otimes t^n \mathbb{C}[t])$$

From a formula by Kac-Kazhdan for the determinant of the Shapovalov form, it follows that the module $V_k(\hat{\mathfrak{g}})$ contains null vectors besides the highest weight vector v_k only when $k = -h^{\vee}$ (the critical level). The space of null vectors $\hat{\mathfrak{z}}(\mathfrak{g})$ of $V := V_{-h^{\vee}}$ is isomorphic to $End_{\hat{\mathfrak{g}}}(V)$. To each vector $v \in V$, we associate a power series:

$$v \longrightarrow Y(v, z) = \sum_m v_m z^m$$

The coefficients of these power series are elements of $\tilde{U}_{-h^{\vee}}(\hat{\mathfrak{g}}) = \tilde{U}(\hat{\mathfrak{g}})/(K + h^{\vee})$. These coefficients span a Lie subalgebra $\tilde{U}_{-h^{\vee}}(\hat{\mathfrak{g}})_{\text{loc}}$. For instance, if $A \in \mathfrak{g}$, we have:

$$A \longrightarrow Y((A \otimes t^{-1})\, v, z) = A(z) = \sum_n (A \otimes t^n)\, z^{-n-1}$$

This shows that $\hat{\mathfrak{g}} \subset \tilde{U}_{-h^{\vee}}(\hat{\mathfrak{g}})_{\text{loc}}$. Let $Z(\hat{\mathfrak{g}})$ denote the center of $\tilde{U}_{-h^{\vee}}(\hat{\mathfrak{g}})_{\text{loc}}$.

We can demonstrate that:

$$x \in \mathfrak{z}(\hat{\mathfrak{g}}) \Leftrightarrow Y(x, z) \in Z(\hat{\mathfrak{g}})$$

Furthermore, every element of $Z(\hat{\mathfrak{g}})$ can be obtained in this form. A prominent example of this is the Casimir element:

$$S = \frac{1}{2} \sum_n (J_a \otimes t^{-1})^2 \in \mathfrak{z}(\hat{\mathfrak{g}})$$

where J_a (a = 1,...,dim $\mathfrak{g}$) is an orthonormal basis of $\mathfrak{g}$ with respect to the invariant bilinear form. The coefficients S_n of the power series:

$$Y(S,z) = \sum_n S_n z^{-n-2} = \frac{1}{2}\sum_a :J_a(z)^2:$$

are called the Sugawara operators and lie in $Z(\hat{\mathfrak{g}})$. Now, let:

$$M_{\chi,k} = U_k(\hat{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{b}}_+)} \mathbb{C}\chi, \qquad \tilde{\mathfrak{b}}_+ = (\mathfrak{b}_+ \otimes 1) \oplus (\mathfrak{g} \otimes t\,\mathbb{C}[[t]])$$

be a Verma module over $\hat{\mathfrak{g}}$, with $\chi \in \mathfrak{h}^*$ and the highest weight vector $v_{\chi,k}$. Let $\mathfrak{g}^L$ denote the Langlands dual of $\mathfrak{g}$, obtained by exchanging the roles of roots and co-roots. According to classical results, the center $Z(\hat{\mathfrak{g}})$ is isomorphic to $W(\mathfrak{g}^L)$, the space of local functionals on $\mathcal{O}\mathfrak{p}(\mathfrak{g}^L)$. Let $\rho \in \mathcal{O}\mathfrak{p}(\mathfrak{g}^L)$ represent a $\mathfrak{g}^L$-oper on the punctured disc. Then, as noted earlier, ρ defines a central character $\tilde{\rho} : Z(\hat{\mathfrak{g}}) \to \mathbb{C}$. We associate the $\hat{\mathfrak{g}}$-module $M_\chi^\rho = M_{\chi,-h^\vee}/\ker\tilde{\rho}$ with the $\mathfrak{g}^L$-oper ρ, [11]. Therefore, we obtain a map:

$$\mathcal{O}\mathfrak{p}(\mathfrak{g}^L) \to \mathrm{Mod}(\hat{\mathfrak{g}}), \to M_\chi^\rho = M_{\chi,-h^\vee}/\ker\tilde{\rho} \tag{26}$$

This map is part of the correspondence between the connections and bg-modules. The interpretation of $\mathfrak{g}^L$-opers as characters on the center of the dual affine algebra $Z(\hat{\mathfrak{g}})$ is central to this framework [17,19,20, 10,18,21].

(v) *Wakimoto modules:*

Our explanation of Wakimoto modules is brief, but it is essential to complete the correspondence. Suppose we are given a linear function $\chi : \mathfrak{h}((t)) \to \mathbb{C}$. We extend it trivially to $\mathfrak{n}((t))$ and obtain a linear function on $\mathfrak{b}_-((t))$, which we also denote by χ. Instead of considering $Ind_{b-((t))}^{\mathfrak{g}((t))}\mathbb{C}\chi$, we focus on the semi-infinite induction [9,20]. The resulting module is a module over the central extension of $\mathfrak{g}((t))$, i.e., over $\hat{\mathfrak{g}}$ at critical level, where the vacuum is annihilated by $t_{\mathfrak{g}}[t] \oplus \mathfrak{n}_-$. The parameters of the module behave not as functionals on $\mathfrak{h}((t))$, but as connections on the H^L-bundle Ω^i (sheaf of i-forms). These are precisely elements of the space $Conn_{\mathfrak{g}}L(D^\times)$ ($D^\times$ is the punctured disc). We obtain a family of smooth representations of $\tilde{U}_{k_c}(\hat{\mathfrak{g}})$ parametrized by $Conn_{\mathfrak{g}}L(D^\times)$. For $\chi \in Conn_{\mathfrak{g}}L(D^\times)$, denote the corresponding Wakimoto module by W_χ. The center $Z_{k_c}(\hat{\mathfrak{g}})$ acts on W_χ via a character. The corresponding point in $Spec\ (Z_{k_c}\hat{\mathfrak{g}}) = \mathcal{O}\mathfrak{p}(\mathfrak{g}^L)\ (\Delta^*)$ is denoted by $\mu(\chi)$. Thus, we obtain a map:

$$\mu : Conn_{\mathfrak{g}^L}(D^\times) \to \mathcal{O}\mathfrak{p}(\mathfrak{g}^L)\,(D^\times) \tag{27}$$

called the Miura transformation. In the context of opers from Section (3), the Miura transformation can be understood as:

$$\partial_t - \begin{pmatrix} 0 & q_1(t) & \cdots & q_{n-1}(t) \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix} dt \to \partial_t - \begin{pmatrix} \chi_1(t) & 0 & \cdots & 0 \\ 1 & \chi_2(t) & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix} dt$$

This corresponds to the following factorization of the differential operator:

$$\partial_t^n - q_1(t)\,\partial_t^{n-1} - \cdots - q_{n-1}(t) = (\partial_t - \chi_1(t)) \cdots (\partial_t - \chi_n(t))$$

The fiber of the Miura transformation over a nilpotent oper is the variety of all Borel subalgebras of $\mathfrak{g}^L$ containing that oper, known as the Springer fiber over the nilpotent oper. For example, the Springer fiber over 0 is the flag variety of $\mathfrak{g}^L$, [8]. The composition of maps in Eqs (26) and (27) gives a map:

$$Conn_{\mathfrak{g}}L(D^\times) \to \mathcal{O}\mathfrak{p}(\mathfrak{g}^L) \to \mathrm{Mod}(\hat{\mathfrak{g}})$$

This provides the correspondence between D-modules of flat connections and $\hat{\mathfrak{g}}$-modules or vertex algebras. The map in the above composition is analogous to the Harish-Chandra homomorphism:

$HC : Z(\mathfrak{g}) \to \mathbb{C}[\mathfrak{p}^*]^W$

A variation of MHS on the punctured disc can be interpreted as a flat connection, regarded as an element of $Conn_{\mathfrak{g}}L(D^{\times})$ via the action of $\mathfrak{g}^L$ arising from the internal symmetries of the Hodge structure. In many cases, the Lie algebra action is paired with a compatible action of the Lie group K to form a Harish-Chandra pair $(\mathfrak{g}, K)$.

(vi) *Geometric Langlands correspondence*:

A more robust way to reformulate Beilinson-Bernstein's localization theorem is through the Geometric Langlands correspondence, which we outline briefly to convey the central idea. For details, refer to [11].

Let X be a smooth projective curve defined over a finite field $\mathbb{F}_q$, and let G be a split connected simple algebraic group also defined over $\mathbb{F}_q$. For a closed point $x \in X$, let $\mathcal{O}_x$ denote the completion of the local ring at x, i.e., $\mathcal{O}_x = \mathbb{F}_q(x)[[t]]$, and let K_x be its field of fractions, where $q_x = q^{\deg x}$.

There exists a correspondence between the conjugacy classes in $G^L(\overline{\mathbb{Q}}_l)$ and G_x-modules, known as the local Langlands correspondence.

The global Langlands correspondence asserts that an irreducible unramified representation $\bigotimes'_x \pi_x$ is automorphic if and only if there exists a continuous homomorphism

$\sigma : \pi_1(X) \to G^L(\overline{\mathbb{Q}}_l)$

such that each π_x corresponds to the conjugacy class $\sigma(Fr_x)$ in the local Langlands correspondence.

Now, let X be an algebraic curve defined over $\mathbb{C}$. Let $G_{\text{in}} = G[[t]]$, and let $\mathcal{O}^0_{\text{crit}}$ be the category of unramified $\hat{\mathfrak{g}}$-modules of critical level. These are modules on which the action of $\mathfrak{g}_{\text{in}} = \mathfrak{g}[[t]]$ is locally finite and contains $\mathfrak{g}_{\text{in}}$-invariant vectors. On such modules, the action of $\mathfrak{g}_{\text{in}}$ can be integrated to an action of the Lie group G_{in}. The analogue of a conjugacy class in G^L is a regular $\mathfrak{g}^L$-oper on the formal disc. The analogue of the local Langlands correspondence is:

Each regular $\mathfrak{g}^L$-oper ρ_x on the formal disc defines an irreducible $\hat{\mathfrak{g}}$ module of critical level.

Suppose we are given a g^L-oper ρ_x for each point $x \in X$. Let V^{ρ_x} be the corresponding $\hat{\mathfrak{g}}$-module of critical level defined in Section (4). Let $\mathfrak{g}(\mathbb{A}) = \prod'_x \mathfrak{g}((t_x))$, and let $\hat{\mathfrak{g}}(\mathbb{A})$ be its one-dimensional central extension. The product $\bigotimes'_x V^{\rho_x}$ is naturally a $\hat{\mathfrak{g}}(\mathbb{A})$-module. One can assign a twisted D-module to the $\hat{\mathfrak{g}}(\mathbb{A})$-module $\otimes_x M_x$ via the localization functor defined earlier:

$\text{loc} : \bigotimes'_x V^{\rho x} \to \Delta(\bigotimes_x M_x)$

The action of $\mathfrak{g}_{in}$ on M_x can be integrated to an action of G_{in}. Therefore, the associated D-module is K-equivariant and descends to a Dmodule on $\mathcal{M}$, denoted by $\Delta(\otimes_x M_x)$. Specializing to $k = -h^{\vee}$, the $\hat{\mathfrak{g}}(\mathbb{A})$-module $\otimes_x V^{\rho_x}$ is called weakly automorphic if $\Delta(\otimes_x V^{\rho_x}) \neq 0$. The weak version of the global Langlands correspondence over $\mathbb{C}$ is as follows:

The $\hat{\mathfrak{g}}(\mathbb{A})$-module $\otimes_x V^{\rho_x}$ is weakly automorphic if and only if there exists a globally defined regular $\mathfrak{g}^L$-oper ρ on X such that for each x, the ρ_x is the restriction of ρ to a small disc around x.

(vii) *Knizhnik-Zamolodchikov* (KZ) *Equations*:

We present this section as a well-known example in conformal field theory. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra with an invariant bilinear form κ. Let $\widehat{\mathfrak{g}_k}$ be the affine Lie algebra with level k and dual Coxeter number $h^{\vee}$. The null vector of a $\widehat{\mathfrak{g}_k}$-module defines the differential equations

$$(k + 2)\frac{\partial}{\partial z_i}\Psi = \sum_{i \neq j}\frac{\Omega_{ij}}{z_i - z_j}\Psi \tag{28}$$

called KZ-equations, where $\Omega_{ij} = \sum_a J^a J_a$ are matrices. J^a and J_a form a dual basis with respect to the invari ant bilinear form κ on $\mathfrak{g}$. If $\mathfrak{g}$ is semisimple, its local system corresponds to representations of the braid group

$$\theta : B_n \to V_1 \otimes \cdots \otimes V_n$$

as the holonomy of the Hamiltonian system (28) (via the RiemannHilbert correspondence or the non-abelian Hodge theorem). A more explicit form of this equation is:

$$dw = \sum_{1 \le i < j \le n} \frac{dz_i - dz_j}{z_i - z_j} A_{ij} w \tag{29}$$

which can be written as the equation of a flat connection $\nabla^{KZ} = d - \Gamma$, where $\Gamma = \sum \frac{dz_i - dz_j}{z_i - z_j} A_{ij}$. In the simplest case of 3-correlations over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, Eq (28) can be reduced to a single-variable equation:

$$\frac{d\phi}{dx} = \left(\frac{A}{x} - \frac{B}{1-x}\right)\phi,\ \phi \in W\{x\}\ [\log x] \tag{30}$$

after an appropriate change of variables, where $A, B \in \mathrm{End}(W)$ for a g-module W, are diagonal matrices. The solution system for Eq (29) can be described in two ways, as follows:

– Suppose $\{\lambda\}$ is the set of eigenvalues of A, where all eigenvalues of A lie in $\cup_\lambda \lambda + \mathbb{N}$. For each λ, there exists a set $\{\lambda + N_j^\lambda\}_{j=0}^{J_\lambda}$ such that $0 = N_0^\lambda < \cdots < N_{J_\lambda}^\lambda$. Let $\pi_\lambda{}^A$ denote the projection onto the λ-eigenspace. A basic but not trivial calculation shows that for any $w \in W$, there is a unique solution to (29) of the form

$$\phi_w^A(x) = \sum_\lambda \sum_j \sum_{i \le N_0^\lambda} w_{i,j}^{(\lambda)} x^{\lambda + i} (\log x)^j$$

where $w_{0,0}^{(\lambda)} = \pi_\lambda^A(w)$, and for each $j > 0$, we have $\pi^A_{\lambda_{N_j^\lambda,0}}(w^{(\lambda)}_{\lambda_{N_j^\lambda,0}}) = \pi^A_{\lambda_{N_j^\lambda,0}}(w)$.

– With a similar setup but focusing on the B-eigenvalues, it can be shown that for any $w \in W$, there is a unique solution of

$$\frac{d\phi}{dy} = \left(\frac{B}{y} - \frac{A}{1-y}\right)\phi,\ \phi \in W\{y\}\ [\log y]$$

of the form

$$\phi_w^B(y) = \sum_\mu \sum_k \sum_{i \ge M_k^\mu} w_{i,j}^{(\mu)} y^{\mu + i} (\log y)^k$$

In the first case, the map

$$\phi_A : \to \phi_w^A(z)$$

defines a $\mathfrak{g}$-isomorphism between W and the solution system of the KZ-equation. In the second case, the map

$$\phi_B : w \to \phi_w^B(1 - z)$$

defines a $\mathfrak{g}$-automorphism $\Phi_{KZ} = \phi^{-}{}_B{}^1 \phi_A$ of W, known as the Drinfield associator of W, which can be interpreted by the intertwining operators describing the tensor structure of the solutions to the KZ-equation [2].

(*viii*) *Conformal blocks:*

We conclude briefly summarizing the concept of conformal blocks, [3]. As mentioned earlier, a vertex algebra is called conformal if it contains the generating function of the basis elements of the Virasoro algebra among its vertex operators. Such algebras or their modules can then be viewed as modules over the Virasoro algebra. For a conformal vertex algebra V, one can associate a vector bundle $\mathcal{V}$ over an algebraic curve, taken as the base manifold X. In this setup, a vertex operator can be interpreted as a section of the dual bundle $\mathcal{V}^*$ on the punctured disc $D_x^\times$, with values in End $(\mathcal{V}_x)$, which can be written as

$$A \otimes z^n dz \to \mathrm{Res}_{z=0} Y(A, z) z^n dz.$$

In the affine Kac-Moody case, given a $\hat{\mathfrak{g}}$-module M_x, we define its space of co-invariants as the quotient $M_x / \mathfrak{g}_{out}(x) \cdot M_x$. The space of conformal blocks is then the dual space:

$(M_x/\mathfrak{g}_{out}(x) \cdot M_x)^\vee = \mathrm{Hom}_{\mathfrak{g}_{out}(x)}(M_x, \mathbb{C})$.

The basis elements of $\hat{\mathfrak{g}}$ are given by $J^a(z)$, and $J^a(z)\,dz$ naturally forms a one-form. A functional $\phi \in M_x^*$ is a conformal block if and only if $\langle \phi, J^a(z) \cdot A\rangle$ has a regular singularity at x.

If we are given a vertex algebra V and a Harish-Chandra pair $(\mathcal{B}, \mathcal{G}_+)$, representing the internal symmetries (for instance, in the case of a Hodge structure derived from the Mumford-Tate group), one can associate a twisted space of co-invariants $H_\tau(X)$ to this data. Here, the Lie algebra action of $\mathcal{B}$ is induced by the Fourier coefficients of the vertex operators in V, while $\mathcal{G}_+$ represents certain symmetries of a geometric data $\mathcal{H}$ (in our case, the local system of Hodge structure) on the punctured disc $D^\times$. As τ varies, the spaces $H_\tau(X)$ combine into a twisted D-module $\Delta(V)$ on the moduli space $\mathcal{M}_\Phi$, which parametrizes the data Φ on X.

5 Conclusion

This article aimed to recollect some facts about Hodge structures and theory of vertex algebras as highest weight modules over affine algebras. We explained several correspondences concerning these structures such as Beilinson-Bernstein correspondence and its generalizations.The idea of inter-relations between these theories already appeared in various areas of mathematics research and in mathematical physics. We studied non-abelian Hodge correspondence, Harish-Chandra correspondence, Beilinson-Bernstein correspondence and geometric Langlands correspondence, as a bridge between integrable systems and representation theory of Lie algebras. These concepts have deep connections to various research areas in physical sciences such as string theory, particle physics, optics and information technology.

Vertex algebras provide a powerful frame work for studying the interplay between representation theory, and theoretical physics. They also provide a robust language for describing conformal field theories, integrable systems, and string theory. This framework also opens doors to exploring dualities and symmetries in quantum field theories and string theory, offering a rigorous mathematical foundation for concepts such as conformal blocks and twisted D-modules.

Representations of affine algebras play a pivotal role in physics, optics, and information technology by providing a robust mathematical framework for describing symmetry, quantization, and state evolution in complex systems. In physics, they are foundational in Conformal Field Theory (CFT), influencing string theory, quantum field theory. In optics, vertex algebras provide tools for photon correlations and entanglement in quantum optics, enabling precise modeling of light propagation in structured media, meta-materials, and topological photonic systems. In information technology, especially in quantum computing and cryptography, vertex and affine algebras contribute to the design of topological quantum codes and robust error-correcting schemes, enhancing quantum information processing.

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