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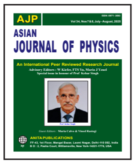

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Alhazen's Paradox Revisited: Unraveling an 11th Century Optical Enigma

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Dedicated to Prof Kehar Singh

This paper, dedicated to honoring Prof Kehar Singh, offers a concise yet insightful overview of the life and scientific contributions of the 11th-century Arab Scholar Alhazen. Renowned for his magnum opus, *The Book of Optics*, Alhazen made ground breaking advancements in the field of optics, including formulating the famous "Alhazen's Problem"—a longstanding paradox in geometrical optics. In this work, we present an unexpected solution to this intriguing problem, shedding new light on its significance and reinforcing Alhazen's enduring legacy in the history of science.

Keywords: History of science, History of optics, Geometrical optics.

1 Introduction

Among the anniversaries considered by the United Nations in declaring 2015 the International Year of Light and Light-Based Technologies, one held particular significance. It marked approximately one thousand years since the publication of the work of the Arab scientist Abū ʿAlī al-Ḥasan ibn al-Ḥasan ibn al-Haytham, better known in Europe by his Latinized name, Alhazen. This remarkable figure, regarded as the father of optics, authored a seminal work: *The Book of Optics* (*Kitab al-Manazir*), a seven-volume treatise. The introduction of these texts into Europe from the 12th century onward led to the widespread dissemination of knowledge on light theories throughout the Western world.

In this section, we examine different aspects of this book, which was first translated from Arabic into Latin in the 12th century and later into English—both from Latin and directly from Arabic. Through this exploration, we analyze Alhazen's theories on light, emphasizing their significance and lasting impact. In particular, we focus on the so-called '*Alhazen's Problem*,' a topic still under discussion today, though with limited progress toward its solution.

The monumental scientific work of ibn al-Haytham (Alhazen) (965, Basra, present-day Iraq – 1039, Cairo, Egypt) contains pioneering and original contributions that remain relevant to this day. His research encompassed fundamental topics such as the theories of light, the laws of optics, and the mechanisms governing the human visual system. Biographical details about his life can now be found on the previous work, published by us in Spanish, with a reasonable level of accuracy and detail, so we will not delve extensively into this aspect [1].

However, it is important to highlight ibn al-Haytham as a Renaissance-like figure, applying this concept retrospectively to his era, well before the European Renaissance. He was an intellectual and

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humanist—a mathematician, astronomer, scientist, and hydraulic engineer—whose vast curiosity and knowledge led him to read (and translate) the works of great Greek thinkers, critically examining and revising their theories. In particular, his legacy was profoundly influenced by the writings of Greek philosophers such as Aristotle, Plato, Euclid, and Ptolemy, who proposed, at times conflicting theories on the mechanisms of human vision. In a previous study, we explored these early theories within the historical context of Ancient Greece, as well as some of Alhazen's most significant experimental designs [2]. In this section, we specifically examine and analyze the first printed editions of *The Book of Optics* in Europe and their subsequent influence on the teaching of optics in prestigious European universities, from the late 12th century to the present day [3].

The seven volumes that make up *The Book of Optics* were undoubtedly completed by Alhazen in Cairo, where he spent most of his adult life. They are estimated to have been written between 1028 and 1038. Additionally, historians and scholars of his work suggest that ibn al-Haytham visited the Iberian Peninsula, which was then under the influence of caliphates such as that of Córdoba. This region played a crucial role in the expansion of Muslim culture in Western Europe during that era [4]. Alhazen's connection to both the intellectual hub of Cairo and the flourishing scholarly environment of Al-Andalus underscores the vast cultural exchanges that shaped scientific progress in medieval times. His contributions to optics not only revolutionized the field but also laid the groundwork for future discoveries in vision, light, and perception.

The first translation of *Kitab al-Manazir* from Arabic into Latin, titled *Perspectiva* (or *De Aspectibus*), was carried out—or at least is currently attributed to—the Italian translator Gerard of Cremona (1114–1187) or other scholars from the Toledo School of Translators, to which he belonged [5]. This work later played a crucial role in shaping the understanding of optics during the European Renaissance.

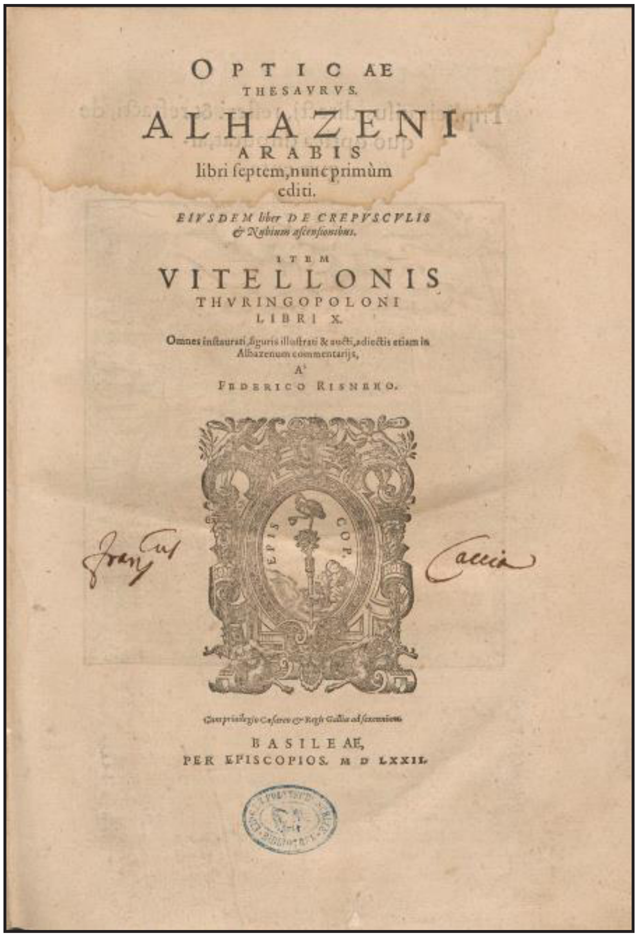
The first European philosopher, scientist, mathematician, and theologian known to have studied and disseminated the works of Alhazen was the English scholar Roger Bacon (1214–1292). Bacon recognized the significance of the Arabic intellectual legacy in philosophy and the sciences, a realization that shaped his interpretation of the laws of optics and the paths of light. His work was grounded in Euclidean plane geometry and closely followed Alhazen's theories.

Another prominent medieval scholar was the Polish philosopher Witelo (1230–1275), also known as Vitellio, a disciple of Albertus Magnus. Deeply interested in Alhazen's work and its transmission across Europe, Witelo produced a handwritten Latin translation from the Arabic, titled *Opticae Thesaurus Alhaceni Arabis*, completed in 1270. Before the later printed editions, other translations of Alhazen's *Treatise on Optics* from Arabic into Latin were completed in the 14th century. Of particular interest is a manuscript published in Rome in 1393 under the title *Liber de Aspectibus et Vocatur Prospectiva*. Originally held by the Casanatense Library, it is now preserved at the Hertzian Library in Germany. The *Treatise on Optics* (*Kitab al-Manazir*) was printed for the first time in Europe in 1572, in the Swiss city of Basel (see Fig 1).

The editor was the German scientist and mathematician Friedrich Risner. Risner edition was entitled as: *Opticae Thesaurus: Alhazeni Arabis Libri Septem, nunc primum editi; Eiusdem liber De Crepusculis et Nubium Ascensionibus*, a Latinized translation of the works of ibn al-Haytham the early pioneer in the field of optics subject of our study [3]. The English translation of the book title is: *Compendium of Optics: The Seven Books of Alhazen the Arab, Published for the First Time; His Book on Twilight and the Ascensions of Clouds.* The second part of the edition, the book on twilight and the ascensions of clouds was erroneously attributed to Alhazen [6].

Risner's publication had a profound impact on the mathematicians and scientists of his time, including Johannes Kepler, Christiaan Huygens, and René Descartes.

In the following section, we will delve deeper into the contents of the *Book of Optics*, exploring Alhazen's meticulous study of light reflection, its trajectories, and its broader implications.



OPTICAE
THESAVRVS.
ALHAZENI
ARABIS
libri ſeptem, nunc primùm
editi.

EIVSDEM liber DE CREPVSCVLIS
& Nubium aſcenſionibus.

ITEM
VITELLONIS
THVRINGOPOLONI
LIBRI X.

Omnes inſtaurati, figuris illuſtrati & aucti, adiectis etiam in
Alhazenum commentarijs,
A'
FEDERICO RISNERO.

Cum priuilegio Cæſareo & Regis Galliæ ad ſexennium.

BASILEAE,
PER EPISCOPIOS. M D LXXII.

Fig 1. Cover of the first printed Latin edition from 1572 of Alhazen's *Book of Optics* (*Kitab al-Manazir*), under the title *Opticae Thesaurus Alhazenis Arabis Libri Septem nunc primum editi. Eiusdem liber De Crepusculis. Nubium Ascensionibus. Item Vitellonis Thuringopolonis Libri X* (see text for full translation).

2 Some relevant aspects of the *Book of Optics* or *Kitab al-Manazir* on theories of light propagation and Associated Phenomena

The preface, signed by Friedrich Risner, consists of three pages. It includes laudatory remarks and a dedication to the regent queen of France, Catherine de' Medici. At the beginning, Risner also states that the present work follows a plan devised by Petrus Ramus his former tutor and professor at the *Collège de France* in Paris.

He describes Alhazen as a distinguished Arab scholar in the field of optics, a discipline that flourished around the year 1000 CE. He also references Alhazen's contemporaries, such as Avicenna, as well as other Arab scientists like the philosopher of Cordovan origin, Averroes, noting their Saracen origins.

Risner further states that he spent an entire year in confinement in Basel to restore the work and acknowledges the significant contributions of Greek philosophers and mathematicians, including Euclid, Ptolemy, and Apollonius.

The preface concludes with a significant statement: *Opticae artis vi ac facultate omnia efficiuntur* ("All things are composed by the power and capability of the science of optics").

It then adds:

"This refers to Alhazen's ancient and prolific teachings on optics, preserved in obscure writings that, after so much time, have been rescued from dust and decay for public scrutiny."

The *Book of Optics*, *Opticae Thesaurus*, is structured as follows:

Liber Primus (First Book) explores vision and light in seven chapters: the nature of light and color perception, the contrast between visible and hidden light, body colors, eye structure, vision quality, optical instruments, and essential visual connections.

Liber Secundus (Second Book) explores vision through three chapters: the arrangement of radial lines and their functions, the perception and understanding of visual entities, and how vision varies in different situations.

Liber Tertius (Third book) examines visual illusions and errors, covering misperceptions, mistaken interpretations, and how vision can be misleading in sensory experience and knowledge.

Liber Quartus (Fourth book) discusses three types of vision—direct, reflected, and refracted—focusing on reflections in polished surfaces, their formation, and how they shape visual perception.

Liber Quintus (Fifth book) consists of two chapters: The first one on Introductory Book and the second one on the locations of images.

Liber Sextus (Sixth Book) consists of nine chapters discussing on accidental errors in the interpretation of reflection, on the errors occurring in plane and convex mirrors and on errors occurring in convex and concave columnar, pyramidal and spherical mirrors.

Liber Septimus (Seventh Book) consists of seven chapters dedicating to the phenomena of light passing and refracting through transparent bodies in straight vertical lines. A study for understanding transparent bodies whose transparency varies, as, for example, when the transparency is oblique to the perpendicular lines of the surface and the illusions of sight produced by refraction.

This extensive work concludes with a length of 282 pages in this 1572 Latin edition, featuring a wealth of illustrations, designed by the editor Risner, to help the reader grasp the subjects discussed in each case. We believe, it holds symbolic significance to present here the final sentences written in Alhazen's *Book of Optics*, *Opticae Thesaurus*, in its 1572 edition, as it reaches the end of its 282 pages:

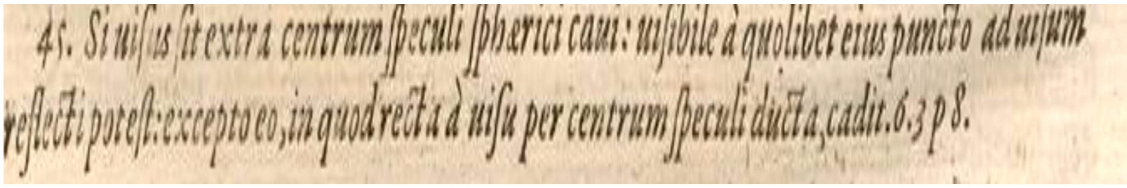
«Nunc autem terminemus hunc tractatum, qui est finis libri»

Which translates as: *"But now this discussion ends, as this book comes to a close."*

3 The Alhazen's problem: an unexpected solution

The so-called *Alhazen's Problem* has long been known to scholars of optics and to mathematicians interested in the history of optics and Euclidean geometry [7]. This problem concerns the phenomenon of light reflection on a spherical mirror and establishes the conditions that the reflected light path must satisfy—considering a single reflection—so that the angles formed at the point of reflection on the spherical surface are equal.

This problem is presented in Alhazen's *Book of Optics* (*Kitab al-Manazir*) and illustrated in a diagram found in Book Four, Chapter Four, Section 45. It is worth noting that Book Four deals with the reflection of light on polished surfaces. Below, we reproduce the title of this section (see Fig 2).



45. Si uiſus ſit extra centrum ſpeculi ſphærici caui: uiſibile à quolibet eius puncto ad uiſum reflecti poteſt: excepto eo, in quod recta à uiſu per centrum ſpeculi ducta, cadit. 6.3 p 8.

Fig 2. Reproduction of the title of Section 45, Chapter Four of Book Four in *Opticae Thesaurus*, Latin edition 1572, which presents the problem of light reflection at a point on the surface of a concave mirror. The study of light trajectories gives rise to the so-called *Alhazen's Problem* (see translation in the text).

Which we transcribe here:

> << *Si visius fit extra centrum speculi sphaerici cavi: visibile à quolibet eius puncto ad visum reflecti potest: excepto eo, in quod recta à visu per centrum speculi ducta, cadit* >>

And whose translation to English is:

<<If vision occurs outside the center of a concave spherical mirror, the visible object can be reflected from any of its points to the eye, except at the point where the straight line drawn from the eye through the center of the mirror falls>>

Figure 3 reproduces an illustration from the *Book of Optics*, corresponding to the text we have translated. The figure depicts several lines representing directions of observation. To clarify its elements, we first identify the key points in the drawing: Letter *b* marks the center of the circle, whose internal surface functions as a concave mirror. Letter *a* represents the point from which light is eventually emitted. Letter *g* indicates the point where light is reflected from the concave mirror. Letter *h* designates the possible location of an observer.

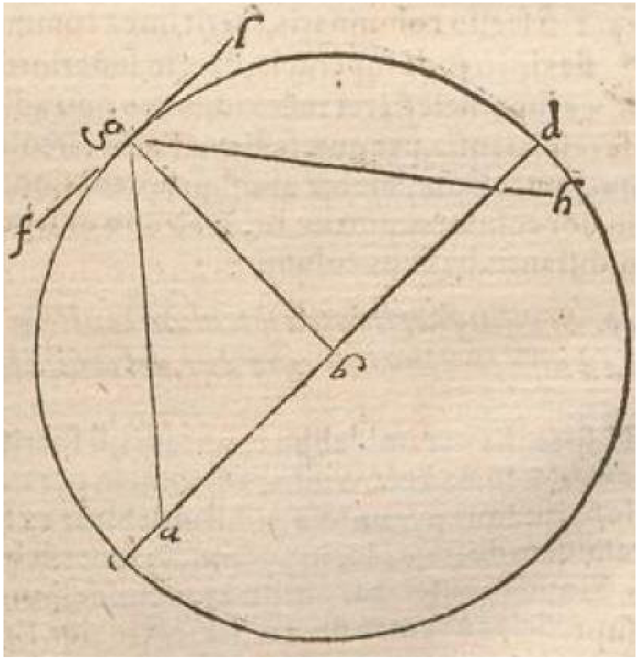


Fig 3. Reproduction of the illustration from *The Book of Optics* (*Opticae Thesaurus*) by Alhazen, as it appears in the 1572 edition. The figure is found in Book Four, Chapter Four, Section 45. It provides a simplified depiction of the so-called "*Alhazen's Problem*." (See text for details).

An arbitrary point *g* is chosen on the surface of the sphere, and a tangent line *fl* is defined at that point. This point *g* represents the point of incidence where light strikes the concave reflecting sphere. The center of the sphere is fully determined and is labeled *b*, where an observer could potentially be positioned. A diameter *cd* is also defined, passing through *b*. The point *a* is located inside the sphere, while *d* lies on its surface.

The point *h* is chosen arbitrarily and, as a general assumption, does not lie on the line $\overline{ad}$. We assume that point *g* is contained within the plane formed by $\overline{agb}$, making this a problem of plane geometry, specifically a two-dimensional problem. Clearly, points *a* and *h* must be inside the sphere. The phenomenon of reflection occurs when a light ray traveling from *a* to *h* (or vice versa) reflects off the surface at point *g*, satisfying the condition that the angles $\widehat{agb}$ and $\widehat{bgh}$ are equal. Alhazen's problem consists of determining the precise point *g* such that the corresponding trajectory satisfies this property, accurately describing the reflection of light rays.

Some scholars have referred to this as *Alhazen's Billiard Problem*, and it is remarkable that, even a thousand years after its formulation, discussions continue on how to find a rigorous solution. Some researchers, such as Peter M Neumann [7], have approached the problem using the equation of a spherical surface, assuming a unit radius and seeking the roots of the equation defined by the three points *a*, *g*, and *h*, respectively. This results in a fourth-degree polynomial. However, the solution obtained is not definitive. Neumann points out an additional issue: the points *a* and *h* are not independent.

2.1. An algebraic solution to Alhazen's problem

We have approached the problem from a strictly geometric perspective and established algebraic equations for both the circle (centered at the origin) and the tangent line at the desired point (where the light is reflected), which lies on the circumference. The problem is greatly simplified by projecting the system onto a plane, as shown in Fig 4. The sphere is projected onto the drawing plane for simplicity. The point $P(x_P, y_P)$ lies on the circumference, whose interior is mirrored, and is coplanar with points *A* and *B*, which are inside the circumference of radius *R* and centered at the origin *O*(*0*, *0*), also for simplicity.

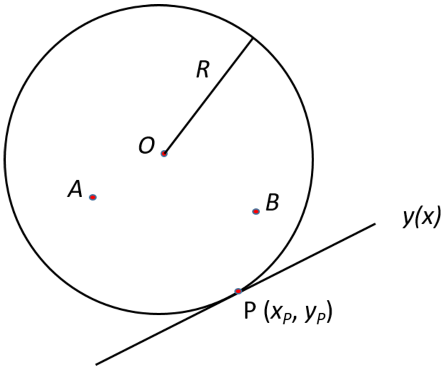


Fig 4. Setup of Alhazen's problem. *P*, *O*, *A*, and *B* correspond, respectively, to *g*, *b*, *a*, and *h* in Fig 3. *P* is initially arbitrary; the one that satisfies the given conditions is the solution to Alhazen's problem.

The problem is to find the coordinates of point P on a circle of radius R (centered at the origin, for simplicity) such that, given points A and B inside the circle, the segment $\overline{OP}$ bisects the angle $\widehat{APB}$.

We assume, for simplicity, that the circle is centered at the origin *O* (0, 0). The equation of the tangent line to a circle at point *P* can be calculated from the circumference equation:

$$x_P x + y_P y = x_P^2 + y_P^2 = R^2 \tag{1}$$

Therefore, $m = -x_P / y_P$ is the slope of the tangent line.

Given a tangent line to a circle of radius R at a point X, the minimum distance from two points A and B to this line (i.e., the sum of both distances) occurs at point M (x_M, y_M), as shown in Fig 5.

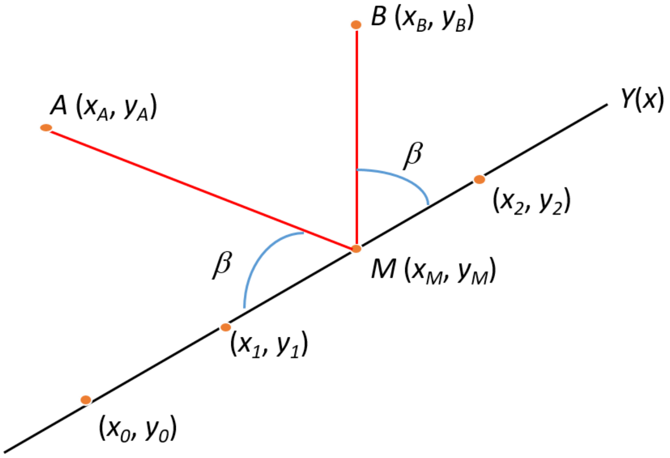


Fig 5. Consider a tangent line (as shown in Figs 3 and 4) and a series of known points, namely, that lie on it. The point M (x_M, y_M), which is unknown and must be determined, is the only one that satisfies the condition for the optical path to be minimal (see text for details).

The tangent line to a circle of radius R at a point B (x_B, y_B) on the circle has a slope equal to the negative reciprocal of the slope of the radius connecting the origin to point B. Then, the slope of the radius at (x_B, y_B) is: $m_{radius} = x_P / y_P$. Therefore, the straight line perpendicular to the anterior tangent line to a circle of radius R, passing by the point (x_B, y_B) is:

$$y(x) = y_B + \frac{y_0}{x_0}(x - x_B) \tag{2}$$

Similarly, the straight line perpendicular to the anterior tangent line to a circle of radius R, passing by the point $A(x_A, y_A)$ is:

$$y(x) = y_A + \frac{y_0}{x_0}(x - x_A) \tag{3}$$

The corresponding points of intersection between each pair of given lines in Eqs (2) and (3), respectively, are:

$$\left(x_0 + \frac{x_B y_0^2 - y_B x_0 y_0}{R^2}, y_0, \frac{y_B x_0^2 - x_B x_0 y_0}{R^2}\right) \equiv (x_2, y_2) \tag{4}$$

and:

$$\left(x_0 + \frac{x_A y_0^2 - y_A x_0 y_0}{R^2}, y_0, \frac{y_A x_0^2 - x_A x_0 y_0}{R^2}\right) \equiv (x_1, y_1) \tag{5}$$

It holds that:

$$\frac{(x_1 - x_M)^2 + (y_1 - y_M)^2}{(x_1 - x_A)^2 + (y_1 - y_A)^2} = \frac{(x_2 - x_M)^2 + (y_2 - y_M)^2}{(x_2 - x_B)^2 + (y_2 - y_B)^2} \tag{6}$$

Together with:

$$y_M = \frac{R^2 - x_0 x_M}{y_0} \tag{7}$$

From the preceding equations, we can express x_M and y_M as functions of (x_0, y_0), (x_1, y_1), and (x_2, y_2), which are all known values.

A very interesting observation about this problem is that the sought point (x_P, y_P), which solves Alhazen's problem, is precisely the one that minimizes the sum of the distances from A to P plus the distance from P to B. That is, if Alhazen's problem is interpreted optically, it corresponds to a minimal optical path problem.

We notice that the problem is greatly simplified by performing a rotation of the system represented in Fig 4, as shown in Fig 6.

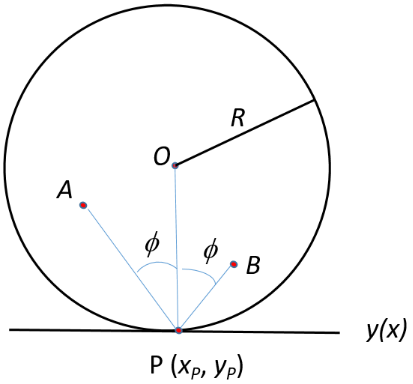


Fig 6. A simple rotation of Fig 4 provides a clearer understanding of the desired solution. The radius bisects the path, defining equal angles ϕ and ensuring the minimum optical path, as predicted by Alhazen.

Let us also note that the angle between $\overline{OP}$ and $\overline{AP}$ is equal to the angle between $\overline{OP}$ and $\overline{PB}$ (optically, for (x_M, y_M) the angles of incidence and reflection are equal).

We have already made a very remarkable observation in this problem: the special point $M\ (x_M, y_M)$, that we have determined and that solves Alhazen's problem is precisely the one that minimizes the sum of the distances from A to P and from P to B. In other words, if Alhazen's problem is interpreted optically, it corresponds to a problem of the minimum optical path. This solution can thus be formulated in terms of Fermat's principle, considering that this principle was formulated by Pierre de Fermat in 1662, more than six centuries after Alhazen originally posed the problem [8]. Under the geometrical optics approximation, the concept of a ray is fundamental for establishing Fermat's principle. Physically, a ray represents the path along which luminous energy propagates in the limit where the wavelength approaches zero. The typical wavelength of visible light is approximately 10^{-7} m, which is small compared to the dimensions of common optical instruments. Consequently, geometrical optics can be applied to a wide range of physical situations, including the one studied here, which involves the reflection of light from a mirror [9].

3 Discussion and Conclusions

We have revisited the so-called *Alhazen's Problem*, also known as the *Billiard Problem*, originally studied and formulated by the Arab Scientist Alhazen in the 11th century. This problem was extensively discussed in his magnum opus, *The Book of Optics*, specifically in the fourth volume, which is dedicated to the phenomena of light refraction and reflection.

Although Alhazen's Problem is most famously associated with his work, its origins can be traced back to the earlier studies of the Greek Mathematician Euclid. Over the centuries, numerous scholars have examined and approached it from different perspectives. For instance, Neumann [7] proposed a solution based on the irreducibility of a fourth-degree polynomial representing the mirrored concave surface. Later, Fujimura *et al* [10] introduced new mathematical proofs using the so-called triangular ratio metric.

In our approach, we have introduced algebraic equations to derive an alternative solution, demonstrating its connection to Fermat's principle of the minimum optical path. If we turn to Lagrangian mechanics, this problem has an elegant and immediate resolution by applying the principle of the minimum optical path—that is, requiring that the derivative of the optical path with respect to the position coordinate be zero. This solution emerges in the most trivial case, when the circle has an infinite radius—in other words, when it is a straight line and thus corresponds to a plane mirror. From this condition, we derive the law of reflection and the law of refraction (better known as Snell's law [11]). These two laws form the fundamental principles for ray tracing, which simply describes the passage of light through an optical system, such as lenses, mirrors, and other optical components.

We conclude, then, that the so-called Alhazen's problem is a problem of minimum time for the transit of light from one internal point to another within a circumference, passing through a point of tangency on the circumference. Only this condition ensures that the line bisects the angle as depicted in Fig 6.

4 References

1. Calvo, M L, The pioneer of Light: Alhazen and his Book of Optics [In Spanish, El Pionero de la Luz: Alhazen y su Libro de la Óptica], (Ediciones Complutense, Madrid, Spain) 2019.
2. Calvo, M L, The exceptional contribution of Ibn-al-Hytham (Alhazen) to science. An attempt to approach the science of optics in the Muslim Renaissance of the 11th C. [In Spanish, La excepcional contribución de ibn-al-Hytham (Alhazen) a las ciencias. Un intento de aproximación a la ciencia de la óptica en el Renacimiento musulmán del S. XI], *Rev Esp Fís*, 29(2015)31–35.
3. Risner F (ed), Compendium of Optics: The Seven Books of Alhazen the Arab, Published for the First Time, Along with His Book on Twilight and the Rising of Clouds. Also, the Ten Books of Witelo of Thuringia [In Latin, Opticae Thesaurus: Alhazeni Arabis Libri Septem nunc primum editi, Eiusdem Liber de Crepusculis et Nubium Asensionibus. Item Vitellonis Thuringopoloni Libri X. Eusebius Episcopius ed eredi di Nicolus Episcopius], (Printed by Eusebius Episcopius and the Heirs of Nicolaus Episcopius, Basel) August 1572.
4. Gorini R, al-Hytham the man of experience. First step in the science of Vision, *J Int Soc Hist Isl Med*, 2(2003) 53–55.
5. For a deeper understanding of the life and work of Gerard of Cremona, we recommend consulting the entry dedicated to him in the Encyclopedia Britannica. https://www.britannica.com/biography/Gerard-of-Cremona?utm_source=chatgpt.com (consulting March 2025).
6. Sabra A I, The authorship of the Liber de Crepusculis, an Eleventh-Century Work on Atmospheric Refraction, *Isis*, 58(1967)77–85.
7. Neumann P M, Reflections on Reflection in a Spherical Mirror, *Am Math Mon*, 105(1998)523–528.
8. Sabra A I, *Theories of Light from Descartes to Newton,* (Cambridge University Press), 1981.
9. Calvo M L, Perez Rios J, Lakshminarayanan V, Dynamic programming applications in optics. In: Mathematical Optics. Classical, Quantum and Computational Methods, eds Lakshminarayana V, Calvo M L, Alieva T, (CRC Press, Boca Raton, Florida), 2013, Chapter 3, 62–65.
10. Fujimura M, Hariri P, Mocanu, M Vuorinen M, The Ptolemy–Alhazen Problem and Spherical Mirror Reflection, *Comput Methods Funct Theory*, 19(2018)135–155.
11. Ghatak A, Optics, (Tata Mc Graw Hill Publishing Co. Lt. 6th edn, New Delhi, India), 2017, Chapter 3.

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